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Inflection points and nonsingular embeddings of surfaces in $\mathbb{R}^5$

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We define asymptotic direction fields on surfaces embedded in $\mathbb{R}^5$ and characterize their critical points as singular points of order 2 of the embedding in Feldman's sense. We show that these fields are globally defined on generically embedded closed surfaces and from this deduce that nonparallelizable surfaces do not admit nonsingular of order two generic embeddings in $\mathbb{R}^5$. April, 2000
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1. INTRODUCTION

The osculating space of order k at a point p of a m-dimensional manifold M in $\mathbb{R}^n$ can be seen as the linear subspace, $T_p^k M$ spanned by the osculating k-spaces of all the curves contained in M passing through p. A smooth map $f : M \to N$ between smooth manifolds M and N is said to be **nondegenerate** or **non singular of order k** if it induces an injective linear map $T_p^k f : T_p^k M \to T_{f(p)}^k N$, $\forall p \in M$. These maps were studied by E.A. Feldman ([4],[5],[6]), who determined the dimensions m, n of the manifolds M and N for which the set of non degenerate of order k embeddings are open and dense in the set of all the embeddings of M in N with the Whitney C^∞-topology and developped several geometrical applications of these methods.

The existence of nondegenerate of order k embeddings from M to N appears to be related to the global geometry of these manifolds. An interesting question arising in this context is that of which surfaces admit nondegenerate of order 2 embeddings in $I\!R^n$. In order that this question makes sense we must consider $n = 5, 6$, for when $n < 5$ there are no such maps, and for $n > 6$, Feldman proved that they form a dense set in $Emb(M, I\!R^n)$.

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We consider here the particular case of $n = 5$. To approach this problem we use the family of height functions induced by an embedding of a surface M in $\mathbb{R}^5$ as the key to define the concept of asymptotic direction on M. The study of the singularities of this family leads to the characterization of the singular points of order 2 of the embedding, on the one hand as the umbilic singularities of height functions on M, and on the other as the singular points of the fields of asymptotic directions on M. We show that these singularities lie, generically, along closed immersed curves in the surface with possibly some connected components reduced to isolated singular points.

We then prove that a surface generically embedded in $\mathbb{R}^5$ admits at least one and at most five globally defined fields of asymptotic directions, and as an immediate consequence of this we get that *A non parallelizable surface generically embedded in* $\mathbb{R}^5$ *must always have singular points of order 2.*

We observe that it is possible to find a nonsingular of order 2 embedding of the 2-sphere (see [3]) in $\mathbb{R}^5$, but a further analysis shows that it is completely made of (corank 1) singularities of infinite codimension of the different height functions. This is an extremely degenerate example from the viewpoint of contacts of the surface with its set of tangent lines (and hence with the tangent hyperplanes containing these lines). Clearly, any perturbation, as small as desired, will produce singularities of order 2. In fact, a deeper interpretation of our results leads to the conclusion that the only way to obtain a surface embedded in $\mathbb{R}^5$ without singular points of order 2 is introducing singularities of infinite codimension of its height functions.

On the other hand, it can be seen ([14]) that stereographic projection takes inflection points of surfaces in S^4 (considered as embedded in $\mathbb{R}^5$) to semi-umbilic points (i.e., points at which the curvature ellipse degenerates into a segment, see [8],[10] of their images in $\mathbb{R}^4$. From this and the above results for generically embedded surfaces in $\mathbb{R}^5$ we can deduce that

Any nonparallelizable closed surface embedded in $\mathbb{R}^4$ *has at least a curve of semi-umbilics.*

There are still many results to be obtained in this direction, and we expect that further analysis of the global behaviour of the singularities of corank 2 of height functions, as well as a deeper study of the generic configurations of the associated fields of asymptotic directions will contribute to this purpose.

2. DEGENERATE DIRECTIONS AND BINORMALS

Consider an embedding of a surface M in 5-space,

$$f : M \to \mathbb{R}^5.$$

The family of height functions on M associated to this embedding is given by

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$$\begin{array}{rcl} \lambda(f) \ : \ M \times S^4 & \longrightarrow & \mathbb{R}^5 \\ (p,v) & \longmapsto & f_v(p) = \langle f(p), v \rangle. \end{array}$$

Clearly, f_v has a singularity at $p \in M$ if and only if v is normal to M at p. It follows from Looijenga's genericity theorem ([9]) that there is a residual set of embeddings in $C^\infty(M, \mathbb{R}^5)$ with the Whitney C^∞-topology, for which the family $\lambda(f)$ is locally stable and thus, for any $v \in N_pM$, the height function f_v has a singularity of one of the following types at p: Morse (A_1), fold (A_2), cusp (A_3), swallowtail (A_4), butterfly (A_5), elliptic umbilic (D_3^+), hyperbolic umbilic (D_3^-) or parabolic umbilic (D_4). The series $\{A_k\}_{k \geq 1}$ is known as the cuspoids family. They represent singularities of corank 1 (corank(f_v)=corank($Hess(f_v)$)), with normal forms $\{x^{k+1}\}_{k \geq 1}$ and $\mathcal{A}$-codimension $k-1$. The $\{D_k^\pm\}_{k \geq 4}$ series is known as the umbilics family. These singularities have corank 2, their normal forms being given by $\{x^2y \pm y^{k-1}\}_{k \geq 4}$ and have $\mathcal{A}$-codimension $k-1$ (see [1]).

A vector $v \in N_pM$ shall be called **degenerate direction** for M provided that p is a non Morse singularity of f_v, that is, a singularity of $\mathcal{A}$-codimension at least 1.

Given a generic embedding $f : M \to \mathbb{R}^5$, we shall characterize the global distribution of its degenerate directions over the surface M in terms of the coefficients of the second fundamental form of f. Let $\{e_1, e_2, e_3, e_4, e_5\}$ be an orthonormal frame in a neighbourhood of a point p in M, such that $\{e_1, e_2\}$ is a tangent frame and $\{e_3, e_4, e_5\}$ is a normal frame in this neighbourhood. The matrix of the second fundamental form of f with respect to this frame is given by

$$\alpha_f(p) = \begin{bmatrix} a_{20} & a_{11} & a_{02} \\ b_{20} & b_{11} & b_{02} \\ c_{20} & c_{11} & c_{02} \end{bmatrix}$$

where $a_{20} = \langle f_{xx}, e_3 \rangle$, $a_{11} = \langle f_{xy}, e_3 \rangle$, $a_{02} = \langle f_{yy}, e_3 \rangle$, $b_{20} = \langle f_{xx}, e_4 \rangle$, $b_{11} = \langle f_{xy}, e_4 \rangle$, $b_{02} = \langle f_{yy}, e_4 \rangle$, $c_{20} = \langle f_{xx}, e_5 \rangle$, $c_{11} = \langle f_{xy}, e_5 \rangle$ and $c_{02} = \langle f_{yy}, e_5 \rangle$.

This vector valued quadratic form induces, for each $p \in M$, a linear map A_p from the normal space,N_pM, of M at p to the space Q of quadratic forms in the variables x and y. If we represent a vector $v \in N_pM$ by its coordinates (v_3, v_4, v_5) with respect to the basis $\{e_3, e_4, e_5\}$, we have

$$A_p(v_3, v_4, v_5) = v_3(d^2f \cdot e_3) + v_4(d^2f \cdot e_4) + v_5(d^2f \cdot e_5).$$

Now, by using the natural identifications (through the basis induced by the above frame), of N_pM and Q with $\mathbb{R}^3$, we can view this as the linear map $A_p : \mathbb{R}^3 \to \mathbb{R}^3$, whose matrix is $\alpha_f(p)$.

Denote by Σ_p the projectivized normal space of $f(M)$ at $f(p)$. That is, Σ_p represents the set of all (nonoriented) normal directions at $f(p)$. Then it can be shown

Lemma 1

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A generic embedding of a surface M in $\mathbb{R}^5$ induces a decomposition of M into subsets $M_i = \{p \in M : rank(\alpha_f(p)) = i\},\ i = 1, 2, 3$

Moreover,

i) If $p \in M_3$ then there is a closed curve of degenerate directions at p.

ii) The subset M_2 is subdivided, in turn, into the following,

$M_2(2) = \{p \in M_2 :$ *there are two lines of degenerate directions in* $\Sigma_p\}$

$M_2(1) = \{p \in M_2 :$ *there is a unique line of degenerate directions in* $\Sigma_p\}$

$M_2(0) = \{p \in M_2 :$ *there is a unique degenerate direction in* $\Sigma_p\}$

iii) If $p \in M_1$ then there is a unique line of degenerate directions in Σ_p.

Proof: Let C represent the cone of degenerate quadratic forms in Q.

i) If $rank(\alpha_f(p)) = 3$, then ImA_p fills the whole Q and hence $A_p^{-1}(C)$ is a closed curve in Σ_p.

ii) If $rank(\alpha_f(p)) = 2$, then ImA_p is a 2-plane and we may have one of the 3 following possibilities according to the relative position of this plane with respect to the cone C.

a) $ImA_p \cap C$ is a pair of intersecting lines. Then $A_p^{-1}(C)$ gives a couple of intersecting planes in $\mathbb{R}^3$, defining a pair of projective lines in Σ_p.

b) $ImA_p \cap C = \{0\}$. In this case $A_p^{-1}(C) = KerA_p$, which defines a unique point in Σ_p.

c) $ImA_p \cap C$ is tangent to C along one of its generatrices. Then $A_p^{-1}(C)$ is a plane in $\mathbb{R}^3$, defining a projective line in Σ_p.

Finally,

iii) If $rank(\alpha_f(p)) = 1$ we can also have 3 possibilities according to the relative position of the line ImA_p with respect to C: lying inside, outside or on the cone. We observe now that convenient transversality conditions on the 2-jet of f eliminate the last possibility. As for the other two we have that $A_p^{-1}(C) = KerA_p$ which defines a projective line in Σ_p in both cases.

Notice also that the case of $rank(\alpha_f(p)) = 0$ cannot appear for a generic embedding f.
∎

Remark: In the proof of the above lemma we observe that when $p \in M_1$ we may have either that the non degenerate direction corresponds to a height function whose quadratic part is of elliptic type ($Im(A_p)$ is inside C), or that it corresponds to a height function whose quadratic part is of hyperbolic type ($Im(A_p)$ is outside C).

Let $\Delta(p) = det\alpha_f(p)$. Observe that $\Delta^{-1}(0) = M_2 \cup M_1$. In the following lemma we analyze the properties of the subset $\Delta^{-1}(0)$ for a generic embedding f.

Lemma 2

For a generic embedding f of a closed surface M in $\mathbb{R}^5$ we have that

a) M_3 is an open subset of M.

b) $\Delta^{-1}(0)$ is a regularly embedded curve at the points $p \in M_2$.

c) The points in M_1 are singularities of Morse type of the function Δ, therefore corresponding to either selfintersections of $\Delta^{-1}(0)$ or isolated points in the open region M_3.

Proof: $M - M_3 = \Delta^{-1}(0)$ and since Δ is a continuous function on M we have that M_3 must be an open region in M.

Let

$$\begin{array}{rcl} f \ : \ \mathbb{R}^2, 0 & \longrightarrow & \mathbb{R}^5 \\ (x,y) & \longmapsto & (x, y, f_1(x,y), f_2(x,y), f_3(x,y)) \end{array}$$

be the local representation of M in the Monge form at a point $p \in M$. In these coordinates $\Delta(p) = f_{1xx}f_{2xy}f_{3yy} - f_{1xy}f_{2xx}f_{3yy} - f_{1xx}f_{2yy}f_{3xy} + f_{1yy}f_{2xx}f_{3xy} + f_{1xy}f_{2yy}f_{3xx} - f_{1yy}f_{2xy}f_{3xx}$. It follows from this expression that, under appropriate transversality conditions on the 3-jet of f, the set $\Delta = 0$ represents a curve possibly with isolated singular points determined by the vanishing of the derivatives of the function Δ. In fact, taking appropriate changes of local coordinates (that preserve the contact of $Im(A_p)$ with the cone C in Q) we can put f in the following normal forms in a neighbourhood of $p \in M$,

If $p \in M_2(2)$, we can put f in the form

$$f(x,y) = (x, y, x^2 + R_1(x,y), y^2 + R_2(x,y), R_3(x,y)),$$

where $R_i \in m^3$, $i = 1, 2, 3$, i.e., all the derivatives of the R_i vanish up to order 3.

If $p \in M_2(0)$, then analogously f can be written as

$$f(x,y) = (x, y, x^2 - y^2 + R_1(x,y), xy + R_2(x,y), R_3(x,y)),$$

For a point $p \in M_2(1)$ we can take,

$$f(x,y) = (x, y, x^2 + R_1(x,y), xy + R_2(x,y), R_3(x,y)),$$

In each one of the above cases it is a simple (but tedious) calculation to verify that under generic conditions on the 3-jet of f at $(0,0)$, the point p is a regular point of $\Delta^{-1}(0)$.

And, finally, if $p \in M_1$, we can represent f in a neighbourhood of p as

$$f(x,y) = (x, y, x^2 \pm y^2 + R_1(x,y), R_2(x,y), R_3(x,y)),$$

and then it follows that $\Delta_x(0,0) = \Delta_y(0,0) = 0$. Again, generic conditions on j^3f guarantee that $Hess\Delta(0,0)$ is non degenerate and thus this singularity is of Morse type. The saddle points are selfintersections of $\Delta^{-1}(0)$ and at the minimum and maximum the (connected component of the) curve $\Delta^{-1}(0)$ reduces to a point. ∎

Remark: It can be deduced from the definition of the set $M_2(1)$, that it is composed of isolated regular points of the curve $\Delta^{-1}(0)$.

A unit vector $v \in \Sigma_p$ shall be called **binormal direction** for M if and only if f_v has a singularity of cusp type or worse (i.e., the $\mathcal{A}$-codimension of f_v is at least 2) at p.

According to what we have exposed before, these singularities may generically be of one of the following types

1) A_3 (cusp) for points p in an open region of M.
2) A_4 (swallowtail) for points p lying along curves in M.

3) A_5 (butterfly) at isolated points of M.
4) D_4 (elliptic and hyperbolic umbilic) for the point p varying along curves in M.
5) D_5 (parabolic umbilic) at isolated points of M.

We call these directions binormal by analogy to the case of curves in $\mathbb{R}^3$, in which the tangent hyperplane, orthogonal to this direction, passing through the point p has higher order of contact with M than "most other" tangent hyperplanes at p (this is due to the fact that the kind of contact of this hyperplane with the surface is given by the singularity type of the height function associated to its normal direction, see [11]).

The genericity conditions for a locally stable family of height functions on a closed surface ensure that at each point of the open and dense subset M_3, the number of binormals must be finite. We shall see in the next section that this number is actually different from zero at every point of M.

Following Feldman, we say that a point p of M is **2-singular** or an **inflection point** whenever the following vectors $\{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}\}$ are linearly dependent. An embedding $f : M \to \mathbb{R}^5$ is said to be **regular of order 2** if there are no 2-singular points in $f(M)$.

Theorem 4
The following conditions are equivalent,
a) A point $p \in M$ is 2-singular.
b) $\Delta(p) = 0$ (i.e., $p \in M_2(0) \cup M_2(1) \cup M_2(2) \cup M_1(1)$).
c) The point $p \in M$ is a singularity of corank 2 for some height function on M.

Proof: Let $f : M \to \mathbb{R}^5$ be given in the Monge form in a neighbourhood of the point p as in the proof of lemma 2 and let $\alpha_f(p)$ be the matrix of the second fundamental form of f with respect to this normal form. We observe that a point $p \in M$ is regular of order 2 if and only if the following matrix has maximal rank ,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & a_{20} & a_{11} & a_{02} \\ 0 & 0 & b_{20} & b_{11} & b_{02} \\ 0 & 0 & c_{20} & c_{11} & c_{02} \end{bmatrix}$$

but this is equivalent to asking that $rank \alpha_f(p) = 3$ and thus we have that a) and b) are equivalent.

Suppose on the other hand that p is a corank 2 singularity for some height function, say $f_v = f_{e_5}$ on M. This implies that the quadratic form $d^2 f \cdot e_5$ (i.e., the second fundamental form in the direction e_5) vanishes at p. But this means that the matrix $\alpha_f(p)$ has a null row and therefore we have that $\Delta(p) = 0$. Hence c) implies b). Conversely, if $\Delta(p) = 0$, then it is not hard to see that by a convenient change of coordinates we can assume that $d^2 f \cdot e_5 = 0$ and thus p is a corank 2 singularity for the height function in the normal direction e_5. ∎

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Remark: For a generic embedding, the singularities of corank 2 for height functions on M can only be of umbilic type D_k, with $k = 4, 5$.

Corollary 5

Given a generic embedding $f : M \to \mathbb{R}^5$, we have that

a) If $p \in M_2$ then there is a unique direction $v \in \Sigma_p$ such that the height function f_v has an umbilic singularity at p. This direction is given by the intersection of the 2 projective lines when $p \in M_2(2)$, and by the only degenerate direction in case that $p \in M_2(0)$.

b) If $p \in M_1$ then it is an umbilic singularity for all the height functions corresponding to all the degenerate directions for M at p.

Proof: As we saw before, $p \in M_2$ if and only if $rank(\alpha_f(p)) = 2$. But this means that $Ker\alpha_f(p)$ is 1-dimensional and this provides the unique direction for which the associated quadratic form vanishes. But this implies that p is a corank 2 singularity for the corresponding height function.

On the other hand, if $p \in M_1$, then $rank(\alpha_f(p)) = 1$ and $\{Ker(\alpha_f(p))\} = 2$, which tells us that there is a whole projective line of degenerate directions, for which the corresponding height functions have a corank 2 singularity. These will be, generically, of type $D_4^{\pm}$. ∎

Remark: Observe that when $p \in M_2(0) \cup M_1$ the degenerate directions coincide with the binormals, whereas when $p \in M_2(2) \cup M_2(1)$ there are degenerate directions which are not binormals for M at p.

In the following result we characterize the different types of inflection points of a surface M in 5-space in terms of the local geometry of the projections of M into convenient hyperplanes. We remind that, as shown in [12] a point in a surface embedded in 4-space can be hyperbolic, parabolic or elliptic according to the existence of two, one or zero binormal directions for M at the point. Inflection points for surfaces in 4-space are corank 2 singularities of height functions on the surface. A more geometrical characterization of these points in terms of the curvature ellipses can be found in [8].

Theorem 6

Let $p \in M$ be an inflection point, so there is some $v \in N_pM$ such that p is an umbilic singularity for the height function f_v. Let H_v be the orthogonal hyperplane to v passing through p and $\pi_v : \mathbb{R}^5 \to H_v$ the orthogonal projection in the direction v. Then the composition $g_v = \pi_v \cdot f : M \to H_v \equiv I\!R^4$ provides a local embedding of M in $\mathbb{R}^4$. Moreover,

i) $p \in M_2(2)$ if and only if p is a hyperbolic point of the surface $g_v(M)$ in $\mathbb{R}^4$.

ii) $p \in M_2(0)$ if and only if p is an elliptic point of $g_v(M)$.

iii) $p \in M_2(1)$ if and only if p is a parabolic point of $g_v(M)$.

iv) $p \in M_1$ if and only if p is an inflection point of $g_v(M)$.

Proof: Since p is an inflection point we have that $rank(\alpha_f(p)) < 3$. Let $v \in N_pM$ such that p is an umbilic singularity for the height function f_v. Then $v \in Ker(\alpha_f(p))$. Observe now that $Ker(\pi_v) =< v >\subset Ker(\alpha_f(p)) = Ker(A_p)$. Hence $A_p : N_pM \to Q$ induces a linear map $\tilde{A}_p : N_pM \cap H_v \to Q$. The subset $\tilde{A}_p^{-1}(C)$ gives the binormal directions of the surface $g_v(M)$ at p. It is not difficult to verify that

i) If $p \in M_2(2)$ then $\tilde{A}_p^{-1}(C) = A_p^{-1}(C) \cap H_v$ consists of 2 distinct binormal directions. So p is a hyperbolic point of $g_v(M)$.

ii) If $p \in M_2(1)$ then $\tilde{A}_p^{-1}(C) = A_p^{-1}(C) \cap H_v$ is a unique binormal direction. So p is a parabolic point of $g_v(M)$.

iii) If $p \in M_2(0)$ then $\tilde{A}_p^{-1}(C) = Ker(A_p) \cap H_v = \{0\}$, which means that there are no binormal directions at p. So p is an elliptic point of $g_v(M)$.

iv) If $p \in M_1$ then $\tilde{A}_p^{-1}(C) = Ker(A_p) \cap H_v$ gives a direction corresponding to a height function of corank 2. So p is an inflection point of $g_v(M)$.

∎

3. ASYMPTOTIC DIRECTIONS.

Given $p \in M$ suppose that $v \in \Sigma_p$ is a binormal direction at v. Then f_v has a degenerate singularity at p and hence $rank(Hess(f_v)) < 2$. Therefore $dim\{ker(Hess(f_v))\} \geq 1$. An **asymptotic direction** at p, associated to the binormal direction v is any unit vector in T_pM that lies in the kernel of the quadratic form given by the hessian of f_v.

Therefore, any binormal direction at p defines some asymptotic direction at p. In fact, since p is a singularity of type $A_k,\ k > 2$ of a height function f_v, then $rank(Hess f_v) = 1$, and thus there is a unique asymptotic direction associated to the binormal v at p. Whereas,if p is a singularity of umbilic type for some height function (i.e., an inflection point), then there is a whole circle of asymptotic directions associated to v at p.

Remark: If b is a binormal direction at p, we have that the hyperplane, H_b, orthogonal to b passing through p has higher order contact with M at p. Now, this contact has place along the elements $\theta \in Ker(Hess(f_b))$, which means that the straight line through p in the direction θ must have higher order of contact with M at p than most tangent lines to M at p. Is in this sense that we call θ **asymptotic** direction. This clearly generalizes the definition of asymptotic directions for surfaces in $\mathbb{R}^4$ (see [12], [13]). It is worth to point out at this stage, that in the last case the concepts of degenerate and binormal directions coincide.

Let v be a degenerate direction at a point p of M_3, so $rank(Hess(f_v)) = 1$, and let θ be a tangent vector in the kernel of the quadratic form $Hess(f_v)(p)$. We denote by γ_θ the normal section of the surface M in the tangent direction θ. That is, γ_θ is a curve in the 4-space $V_\theta = \langle\theta\rangle \oplus N_pM$, obtained as the intersection of this 4-space with M. The restriction of the family of height functions $\lambda(f)$ to (some parametrization of) the curve γ_θ gives us the family of height functions on this curve. Now, if we take into account that the binormal (or 3rd normal vector) n_3 of a curve in 4-space can be characterized by the fact that the height function over the curve, corresponding to the direction n_3 has a singularity of type $A_k,\ k \geq 3$ (a proof of a similar assertion for curves in 3-space can be found in [2], the case of curves in 4-space is easily proven by using analogous arguments) we obtain the following geometrical characterization of the asymptotic directions at regular points of surfaces in 5-space,

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Proposition 7
Let $p \in M_3$ and $v \in N_pM$ a degenerate direction for M at p. Let θ be a tangent direction contained in $Ker(Hess(f_v)(p))$. Then θ is an asymptotic direction (corresponding to the binormal v) if and only if v is the binormal direction at p for the curve γ_θ in the 4-space V_θ.

Theorem 8
There are at least one and at most 5 asymptotic directions at each point of M_3.

Proof: Let $f(x,y) = (x, y, f_1(x,y), f_2(x,y), f_3(x,y))$ be given in the Monge form, with $f_i(x,y) = q_i(x,y) + k_i(x,y) + R_i(x,y)$, where q_i are quadratic forms, k_i are cubic forms and $R_i \in m^4$, $i = 1,2,3$. Since $p \in M_3$, we have that the 3 quadratic forms $q_1(x,y), q_2(x,y)$ and $q_3(x,y)$ must be linearly independent (in Q) and without loss of generality we can take local coordinates at p in such a way that $q_3(x,y) = -(x^2+y^2)$ and $k_3(x,y) = 0$.

Let $v = (0,0,v_3,v_4,v_5) \in N_pM$ be a solution of the equation $A_p(v_3,v_4,v_5) = 0$. Then $Hess(f_v(p))$ is a degenerate quadratic form and so there is $u \in T_pM$ such that $Hess(f_v(p))(u,w) = 0, \forall w \in T_pM$. That is, $v_3q_1(u,w) + v_4q_4(u,w) - v_5\langle u, \rangle w$. This expresion must be true in particular for the vector u and a vector $v \in T_pM$ orthogonal to u, so we have the expressions,

$$v_3q_1(u,u) + v_4q_4(u,u) - v_5(u^2+w^2) = 0 \quad (1)$$

$$v_3q_1(u,v) + v_4q_4(u,v) = 0 \quad (2)$$

On the other hand, p will be a singular point of cusp type or worse if the vector u satisfies $v_3k_1(u)^3 + v_4k_2(u)^3 + v_5k_3(u)^3 = 0$ (see [10]). And taking into account that in the chosen local coordinates $k_3(x,y) = 0$, this give

$$v_3k_1(u,v) + v_4k_4(u,v) = 0 \quad (3)$$

Once given v_3 and v_4 we can obtain v_5 from (1). On the other hand, eliminating v_3 and v_4 between (2) and (3) gives

$$k_1(u^3)q_2(u,v) - k_2(u^3)q_1(u,v) = 0$$

which leads to a quintic form in u (for being v any vector orthogonal to u in T_pM, its coordinates are linearly given in terms of that of u). This gives the differential equation for the asymptotic lines in M.

We observe that this equation cannot be identically zero on the regular points of order 2 of 43a generic surface. ∎

Corollary 9
For a generically embedded surface M in $\mathbb{R}^5$ there are at least 1 and at most 5 globally defined fields of asymptotic directions on M, whose singularities occur on the curve $\Delta^{-1}(0)$.

Now, from the well known fact that non parallelizable surfaces do not admit globally defined vector fields without critical points we conclude:

Theorem 10
A generic embedding of a nonparallelizable surface in $\mathbb{R}^5$ *cannot be regular of order 2.*

An example of a 2-regular embedding of the 2-sphere in 5-space is given by the restriction to S^2 of the Veronese map of order 2 $V : \mathbb{R}^3 \to \mathbb{R}^6$, given by $V(x,y,z) = (x^2, y^2, z^2, \sqrt{2}xy, \sqrt{2}xz, \sqrt{2}yz)$ ([3]). Studying the singularities of the family of height functions on this surface leads to the conclusion that all their degenerate singularities are of infinite codimension.

4. STEREOGRAPHIC PROJECTION AND SURFACES IN $\mathbb{R}^4$.

Given a surface M in $\mathbb{R}^4$ we define a osculating hypersphere of M at a point p as a hypersphere whose order of contact with M at p is at least 3. This order of contact can be measured through the distance squared function from the center of the given hypersphere (see [11]). Having order of contact k means that this function has a singularity of codimension $k-1$, generically of type A_k or D_{k-1}, at the contact point. Semiumbilic points of M are those at which some distance squared function has a singularity of corank 2. These are generically of type $D_4^{\pm}$ along curves in M and D_5 on isolated points of these curves ([10]). In fact, Montaldi proved that the semiumbilic points of a generically embedded surface in $\mathbb{R}^4$ lie along curves on the surface. He saw that the possible singularities of these curves correspond to the vanishing of the second fundamental form of the embedding (i. e., all the height functions at the point have zero quadratic part). Since this cannot happen for a generic embedding of surface in $\mathbb{R}^4$, he concluded that these curves are smoothly embedded in the surface.

Composing an embedding $g : M \to \mathbb{R}^4$, of a surface in 4-space with the inverse of the stereographic projection leads to another embedding of the surface M in 5-space,

$$f : M \to \mathbb{R}^4 \to S^4 \to \mathbb{R}^5$$

where $\psi : \mathbb{R}^4 \to S^4$ denotes the inverse of the stereographic projection, and the map in the right hand side is the natural inclusion of S^4 in $\mathbb{R}^5$. Now, since the stereographic projection transforms hyperspheres of $\mathbb{R}^4$ into hypercircles of S^4 preserving their respective contacts with $g(M)$ and $f(M)$, it can be seen that any osculating hypersphere having a given contact with $g(M)$ is in correspondence, through ψ, with an osculating hyperplane having the same contact with $f(M)$ in $\mathbb{R}^5$ ([13]). Consequently the semiumbilic curves of surfaces in $\mathbb{R}^4$ are mapped by ψ into our inflection points curves for the corresponding surface in S^4. As we proved in lemma 2, the inflection points of a generic surface M in $\mathbb{R}^5$ are curves with possible singularities at points of type M_1. As a consequence of the above considerations we get that for generic surfaces lying on the S^4, the subset M_1 must be empty, and thus

Corollary 11
If M is a generically embedded surface in S^4 its singular points of order 2 lie along smooth closed curves.

We observe that surfaces contained in S^4 are a special case of convex surfaces, in the sense that they lie on the boundary of their convex hull in 5-space. In view of this we formulate the following

Conjecture: *The singular points of order 2 of a generic convexly embedded surface in $\mathbb{R}^5$ form smooth closed curves.*

Finally, a consequence of the theorem 10 and the fact that stereographic projection takes semiumbilic points of surfaces in 4-space into singular points of order 2 of surfaces in 5-space, is the following

Corollary 12
Any nonparallelizable surface generically embedded in $\mathbb{R}^4$ has at least a closed curve of semiumbilic points.

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