

## Topological Rigidity for Certain Families of Differentiable Plane Curves

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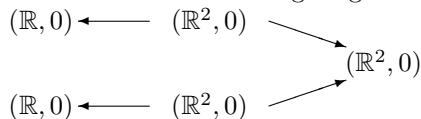
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We show that the topological classification and the smooth classification are generically the same for certain families of plane curves in a semi-local case (the double local case). Especially we give the normal form of transversely jointed two families of plane curves with second order contact at the envelope. April, 2000 ICMC-USP

### 1. INTRODUCTION

The divergent diagrams of map germs  $(\mathbb{R}, 0) \leftarrow (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$  appear in several subjects in geometry (e.g. [1, 2, 4, 6, 7, 8, 10, 13]). For these subjects, in any case such divergent diagrams are identified up to smooth coordinate changes in each spaces. Particularly in [1, 4] a generic classification of such divergent diagrams have been studied. Moreover in [4] it was shown that their topological classification and their  $C^\infty$ -classification are generically the same. Such property is called *topological rigidity*. This classification have been applied to the vision theory ([6, 9, 10]).

In a semi-local case of the above studies the following diagram of map germs appear:



Generic types of such diagrams were studied in [6, 10] and certain normal forms were given in [10] by using the standard method of singularity theory. However their topological rigidity is unsolved problem. In this paper we discuss the following problem.

The two diagrams of  $C^\infty$ -map germs  $(f_1^j, \gamma_1^j; f_2^j, \gamma_2^j), j = 1, 2$  :

$$(\mathbb{R}, 0) \xleftarrow{f_1^j} (\mathbb{R}^2, 0) \xrightarrow{\gamma_1^j} (\mathbb{R}^2, 0) \xleftarrow{\gamma_2^j} (\mathbb{R}^2, 0) \xrightarrow{f_2^j} (\mathbb{R}, 0)$$

are called  $C^\infty$ -(resp. *topologically*) *equivalent* if there exists  $C^\infty$ -diffeomorphism germs (resp. homeomorphism germs)  $h_i : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ ,  $H_i : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ ,  $i = 1, 2$  and  $K : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$  such that  $h_i \circ f_i^1 = f_i^2 \circ H_i$ ,  $K \circ \gamma_i^1 = \gamma_i^2 \circ H_i$  for  $i = 1, 2$ .

*Topological Rigidity Problem* : For the diagrams of map germs

$$(\mathbb{R}, 0) \leftarrow (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0) \leftarrow (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$$

the topological classification and the  $C^\infty$ -classification are generically the same or not?

Our result states that the answer of this problem is affirmative. More precisely to state our theorem we shall recall some results in the following section.

## 2. PREVIOUS RESULTS AND STATEMENTS OF THEOREMS

We shall recall some results of families of plane curves in [1, 3, 4, 6, 10, 14]. Denote by  $\mathcal{E}_{x_1, \dots, x_n}$  (or briefly  $\mathcal{E}_n$ ) the ring of all smooth function germs on  $\mathbb{R}^n$  at 0 with coordinates  $(x_1, \dots, x_n)$  and denote by  $\mathcal{M}_{x_1, \dots, x_n}$  (or briefly  $\mathcal{M}_n$ ) the unique maximal ideal of  $\mathcal{E}_{x_1, \dots, x_n}$ . Denote by  $S_f$  the singular set of a  $C^\infty$ -map germ  $f : (\mathbb{R}^n, 0) \rightarrow \mathbb{R}^m$ . We shall suppose that all map germs are of class  $C^\infty$  unless otherwise stated.

In [4] the generic type of  $(\mathbb{R}, 0) \leftarrow (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$  have been given as follows:

- (I)  $f$  is a submersion and  $\gamma$  is regular.
- (II)  $f$  is of Morse type and  $\gamma$  is regular.
- (III)  $f$  is a submersion,  $\gamma$  is a fold,  $f$  restricted to the singular set  $S_\gamma$  of  $\gamma$  is regular and  $(f, \gamma) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$  is regular.
- (IV)  $f$  is a submersion,  $\gamma$  is a fold,  $f|_{S_\gamma}$  is of Morse type and  $(f, \gamma) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$  is regular.
- (V)  $f$  is a submersion,  $\gamma$  is a fold,  $(f, \gamma) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$  is a Whitney umbrella whose line of double points is transversal at 0 to the direction  $\{0\} \times \mathbb{R}^2$  in  $\mathbb{R}^3$ .
- (VI)  $f$  is a submersion,  $\gamma$  is a cusp and  $(f, \gamma) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$  is regular.

In the semi-local case generic types have been given as follows([6, 10]):

- $(I, I)^0, (I, I)^1, (I, I)^2 : (f_1, \gamma_1), (f_2, \gamma_2)$  are both of type (I) and  $(f_1 \circ \gamma_1^{-1}, f_2 \circ \gamma_2^{-1}) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$  is respectively regular, a fold, a cusp.
- $(II, I) : (f_1, \gamma_1)$  is of type (II),  $(f_2, \gamma_2)$  is of type (I) and  $(f_1 \circ \gamma_1^{-1}, f_2 \circ \gamma_2^{-1}) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$  is a fold.
- $(III, I)^0, (III, I)^1 : (f_1, \gamma_1)$  is of type (III),  $(f_2, \gamma_2)$  is of type (I) and respectively  $\gamma_1(S_{\gamma_1}) \bar{\cap} \gamma_2(f_2^{-1}(0)), \gamma_1(S_{\gamma_1}) \bar{\not\cap} \gamma_2(f_2^{-1}(0))$  with two point contact.
- $(IV, I) : (f_1, \gamma_1)$  is of type (IV),  $(f_2, \gamma_2)$  is of type (I),  $\gamma_1(S_{\gamma_1}) \bar{\cap} \gamma_2(f_2^{-1}(0))$  and  $\bigcup_{t \in (\mathbb{R}, 0)} \gamma_1(S_{\lambda_t}) \bar{\cap} \gamma_2(f_2^{-1}(0))$  where  $\lambda_t \in \mathcal{E}_1$  such that  $graph(\lambda_t) = f_1^{-1}(t)$ .
- $(V, I) : (f_1, \gamma_1)$  is of type (V),  $(f_2, \gamma_2)$  is of type (I),  $\gamma_1(S_{\gamma_1}) \bar{\cap} \gamma_2(f_2^{-1}(0))$ ,  $\bigcup_{t \in (\mathbb{R}, 0)} \gamma_1(S_{\lambda_t}) \bar{\cap} \gamma_2(f_2^{-1}(0))$  where  $\lambda_t \in \mathcal{E}_1$  such that  $graph(\lambda_t) = f_1^{-1}(t)$  and the tangent cone of  $\gamma_1(f_1^{-1}(0)) \bar{\cap} \gamma_2(f_2^{-1}(0))$ .

(VI, I) :  $(f_1, \gamma_1)$  is of type (VI),  $(f_2, \gamma_2)$  is of type (I) and the tangent cone of  $\gamma_1(S_{\gamma_1}) \bar{\cap} \gamma_2(f_2^{-1}(0))$ .

(III, III) :  $(f_1, \gamma_1), (f_2, \gamma_2)$  are both of type (III) and  $\gamma_1(S_{\gamma_1}) \bar{\cap} \gamma_2(S_{\gamma_2})$ .

REMARK 2.2.1. *In case (IV, I) a condition  $\bigcup_{t \in (\mathbb{R}, 0)} \gamma_1(S_{\lambda_t}) \bar{\cap} \gamma_2(f_2^{-1}(0))$  is added to the previous generic condition of (IV, I) in [10]. Even if this condition is added, the above condition of (IV, I) is generic. Similarly in case (V, I) two conditions such that  $\bigcup_{t \in (\mathbb{R}, 0)} \gamma_1(S_{\lambda_t}) \bar{\cap} \gamma_2(f_2^{-1}(0))$  and  $\gamma_2(f_2^{-1}(0)) \bar{\cap}$  the tangent cone of  $\gamma_1(f_1^{-1}(0))$  are added in this paper. These added conditions are effective to solve the rigidity problem.*

Normal forms of the generic types have been given in [1, 3, 4, 10, 14](See sect.5). However we give another normal form for type (III, III) which is more precise than the previous normal form([10]) and moreover effective to show the rigidity problem.

**Theorem A.** *The normal form for type (III, III) is the following:*

$$\begin{aligned} f_1(x_1, y_1) &= x_1 + y_1, \gamma_1 = (x_1, y_1^2) \\ f_2(x_2, y_2) &= x_2 + y_2 + \theta(x_2, y_2), \gamma_2 = (x_2^2, y_2) \end{aligned}$$

where  $\theta$  is an arbitrary function germ in  $\mathcal{M}_{x_2, y_2}$  satisfying  $\theta(x_2, 0) = \theta(0, y_2) = 0$ .

REMARK 2.2.2. *In [10] normal form of (III, III) was given as follows:  $f_1 = y_1 + \alpha_1 \circ \gamma_1$ ,  $\gamma_1 = (x_1, y_1^2)$ ;  $f_2 = x_2 + \alpha_2 \circ \gamma_2$ ,  $\gamma_2 = (x_2^2, y_2)$ , where  $\alpha_1, \alpha_2 \in \mathcal{M}_{u, v}$  with  $\frac{\partial \alpha_1}{\partial u}(0) \neq 0$ ,  $\frac{\partial \alpha_2}{\partial v}(0) \neq 0$ .*

REMARK 2.2.3. *In the semi-local case we remark that the followings: (1) The generic types are not topologically equivalent one another. (2) The normal forms depend on arbitrary functions with some conditions, that is so-called “functional moduli” appear in the normal forms except  $(I, I)^0$ ,  $(I, I)^1$ ,  $(I, I)^2$  and  $(II, I)$ . (On the normal forms of each generic types, see section 5)*

Therefore to solve the topological rigidity problem it is sufficient to consider each generic types with functional moduli.

**Theorem B.** *The topological rigidity holds for every generic types.*

### 3. PROOF OF THEOREM A

Firstly we detect the normal form for  $\gamma_1$  and  $\gamma_2$ . Next we shall find out certain coordinate transformations preserving  $\gamma_1$  and  $\gamma_2$  which give the normal form in Theorem A.

LEMMA 3.3.1. *Let  $(\mathbb{R}^2, 0) \xrightarrow{\gamma_1} (\mathbb{R}^2, 0) \xleftarrow{\gamma_2} (\mathbb{R}^2, 0)$  be a bi-germ such that  $\gamma_1, \gamma_2$  are both fold map germs and that the discriminant sets of  $\gamma_1, \gamma_2$  are transversal at 0 each*

other. Then we can express the bi-germ for some coordinates as follows:

$$\gamma_1(x_1, y_1) = (x_1, y_1^2), \gamma_2(x_2, y_2) = (x_2^2, y_2).$$

**Proof.** We choose coordinates  $(x_1, y_1)$  in the source space of  $\gamma_1$ ,  $(u, v)$  in the target space of  $\gamma_1$  and  $\gamma_2$  such that  $\gamma_1(x_1, y_1) = (x_1, y_1^2)$ . Then we can take coordinates  $(x_2, y_2)$  such that  $\gamma_2(x_2, y_2) = (u \circ \gamma_2, y_2)$ . From the Malgrange preparation theorem, there exist  $A, B \in \mathcal{E}_{u,v}$  such that  $x_2^2 = A \circ \gamma_2 + 2(B \circ \gamma_2)x_2$ . Consider coordinate transformations  $\phi_1, \psi, \phi_2$  which preserve  $\gamma_1$  by  $\phi_1(x_1, y_1) = (A \circ \gamma_1(x_1, y_1) + B^2 \circ \gamma_1(x_1, y_1), y_1)$ ,  $\psi(u, v) = (A(u, v) + B(u, v)^2, v)$ ,  $\phi_2(x_2, y_2) = (x_2 - B(u \circ \gamma_2(x_2, y_2), y_2), y_2)$ . By these coordinate changes we obtain the normal form of the bi-germ. ■

From now on we suppose that in this section always the bi-germ

$$(\mathbb{R}^2, 0) \xrightarrow{\gamma_1} (\mathbb{R}^2, 0) \xleftarrow{\gamma_2} (\mathbb{R}^2, 0)$$

is defined by  $\gamma_1(x_1, y_1) = (x_1, y_1^2)$ ,  $\gamma_2(x_2, y_2) = (x_2^2, y_2)$ .

The following Lemma is obvious:

LEMMA 3.3.2. Let  $(\mathbb{R}^2, 0) \xrightarrow{\gamma_1} (\mathbb{R}^2, 0) \xleftarrow{\gamma_2} (\mathbb{R}^2, 0)$  be a bi-germ given by the above. Let  $H_1, H_2, K : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$  be diffeomorphism germs defined by

$$\begin{aligned} H_1(x_1, y_1) &= (x_1 A^2 \circ \gamma_1, y_1 B \circ \gamma_1), \\ H_2(x_2, y_2) &= (x_2 A \circ \gamma_2, y_2 B^2 \circ \gamma_2), \\ K(u, v) &= (u A^2, v B^2), \end{aligned}$$

where  $A, B \in \mathcal{E}_{u,v}$  with  $A(0) \neq 0, B(0) \neq 0$ . Then  $\gamma_1 \circ H_1 = K \circ \gamma_1$  and  $\gamma_2 \circ H_2 = K \circ \gamma_2$  hold.

We say that the triple of the above diffeomorphism germ  $(H_1, K, H_2)$  is a  $(\gamma_1, \gamma_2)$ -compatible diffeomorphism germ.

LEMMA 3.3.3. The germs of type (III, III) are equivalent to

$$(\mathbb{R}, 0) \xleftarrow{x+y} (\mathbb{R}^2, 0) \xrightarrow{\gamma_1} (\mathbb{R}^2, 0) \xleftarrow{\gamma_2} (\mathbb{R}^2, 0) \xrightarrow{f} (\mathbb{R}, 0),$$

where  $f(x, y) = x + y + \text{higher term}$ .

**Proof.** Let  $(f_1, \gamma_1; f_2, \gamma_2)$  be type (III, III). Firstly we shall show that the type (III, III) is equivalent to  $(x_1 + y_1, \gamma_1; f, \gamma_2)$ , where  $f \in \mathcal{M}_{x,y}$  with  $\frac{\partial f}{\partial x}(0) \frac{\partial f}{\partial y}(0) \neq 0$ . Since  $(f_1, \gamma_1)$  is of type (III),  $\frac{\partial f_1}{\partial y_1}(0) \neq 0$ . So by using the inverse function of  $f_1(0, y_1)$  as a coordinate change in the target of  $f_1$ , we can suppose that  $f_1(0, y_1) = y_1$ . Then by the Malgrange preparation theorem, there exist  $\alpha, \beta \in \mathcal{E}_2$  such that  $f_1(x_1, y_1) = \alpha(x_1, y_1^2) + y_1 \beta(x_1, y_1^2)$ .

Thus  $\alpha(0, y_1^2) = 0$ . Moreover since  $f_1|_{S_{\gamma_1}}$  is non-singular we may suppose that  $\frac{\partial \alpha}{\partial u}(0) > 0$ . Hence we have  $\alpha(x_1, y_1^2) = x_1 \tilde{\alpha}(x_1, y_1^2)$  for some  $\tilde{\alpha} \in \mathcal{E}_2$  with  $\tilde{\alpha}(0) > 0$ . Now set  $A = \sqrt{\tilde{\alpha}}, B = \beta$ . Then using the  $(\gamma_1, \gamma_2)$ -compatible diffeomorphism given in Lemma 3.2 we obtain the required form, where  $f = f_2 \circ H_2^{-1}$ .

Next put  $a = (\frac{\partial f}{\partial x}(0)/\frac{\partial f}{\partial y}(0))^{-\frac{2}{3}}, b = (\frac{\partial f}{\partial y}(0))^{-1}(\frac{\partial f}{\partial x}(0)/\frac{\partial f}{\partial y}(0))^{-\frac{4}{3}}$ . Then consider the coordinate transformations  $h_1(t_1) = at_1$  in the target of  $f_1 = x_1 + y_1, k(t) = bt$  in the target of  $f, (\gamma_1, \gamma_2)$ -compatible diffeomorphisms  $H_1, K, H_2$  by putting  $A = \sqrt{a}, B = a$  in Lemma 3.2. Using these coordinate transformations we obtain the required form in this Lemma.  $\blacksquare$

In order to find coordinate transformations which reduce to the normal form in Theorem A, we now need the following observation. Suppose that the diagram (3.1)

$$\begin{array}{ccccccccc}
 (\mathbb{R}, 0) & \xleftarrow{x+y} & (\mathbb{R}^2, 0) & \xrightarrow{\gamma_1} & (\mathbb{R}^2, 0) & \xleftarrow{\gamma_2} & (\mathbb{R}^2, 0) & \xrightarrow{f} & (\mathbb{R}, 0) \\
 h \downarrow & & H_1 \downarrow & & K \downarrow & & \downarrow H_2 & & \downarrow k \\
 (\mathbb{R}, 0) & \xleftarrow{X+Y} & (\mathbb{R}^2, 0) & \xrightarrow{\gamma_1} & (\mathbb{R}^2, 0) & \xleftarrow{\gamma_2} & (\mathbb{R}^2, 0) & \xrightarrow{\tilde{f}} & (\mathbb{R}, 0)
 \end{array}$$

commutes for some diffeomorphism germs  $h, H_1, K, H_2, k$ , where  $H_1, K, H_2$  have the forms in Lemma 3.2,  $f = x + y + \text{higher term}, \tilde{f}$  satisfies (3.2)

$$\tilde{f}(x, 0) = \tilde{f}(0, x) = x.$$

Then from (3.1) we have  $h(x+y) = xA^2(x, y^2) + yB(x, y^2), k \circ f(x, y) = \tilde{f}(xA \circ \gamma_2(x, y), yB^2 \circ \gamma_2(x, y))$ . Thus we have  $h(x) = xA^2(x, 0)$  and  $h(y) = yB(0, y^2)$ . Therefore  $h$  is increasing and odd. Namely there exists  $a \in \mathcal{E}_1$  with  $a(0) > 0$  such that

$$h(t) = ta(t^2).$$

Hence  $A(u, 0) = \pm\sqrt{a(u^2)}$  and  $B(0, v) = a(v)$  for any  $v \geq 0$ . Moreover from (3.2) we have (3.3)

$$k \circ f(x, 0) = xA(x^2, 0), k \circ f(0, y) = yB^2(0, y).$$

Therefore we have  $k \circ f(x, 0) = \pm x\sqrt{a(x^4)}$  and  $k \circ f(0, y) = ya^2(y)$  for  $y \geq 0$ .

So we now suppose that  $k \circ f(t, 0) = t\sqrt{a(t^4)}, k \circ f(0, t) = ta^2(t)$  hold for all  $t \in (\mathbb{R}, 0)$ . Then we have (3.4)

$$k = t\sqrt{a(t^4)} \circ f(t, 0)^{-1} = ta^2(t) \circ f(0, t)^{-1},$$

where  $f(t, 0)^{-1}, f(0, t)^{-1}$  respectively denoting the inverse function germ of  $f(t, 0), f(0, t)$ . We can write

$$f(0, t)^{-1} \circ f(t, 0) = tb(t)$$

for some  $b \in \mathcal{E}_1$  with  $b(0) = 1$ . Then from (3.4) we have

(3.5)

$$a(t^4) = b^2(t)a^4(tb(t)).$$

Conversely the following holds.

**PROPOSITION 3.3.4.** *If for any  $f \in \mathcal{E}_2$  with the form  $f(x, y) = x + y + \text{higher term}$ , there exists  $a \in \mathcal{E}_1$  with  $a(0) > 0$  satisfying the equation (3.5), then there exists coordinate changes  $h, H_1, K, H_2, k$  such that the diagram (3.1) commutes and  $\tilde{f} = k \circ f \circ H_2^{-1}$  satisfying the property (3.2).*

**Proof.** For  $f = x + y + \text{higher term}$ , let  $a \in \mathcal{E}_1$  with  $a(0) > 0$  be a solution of the equation (3.5). Then from (3.5) we have the equality on the right hand side in (3.4). So that we define  $k(t)$  by (3.4). Now we put  $h(t) = ta(t^2)$ . Then from the Malgrange preparation theorem  $h(x + y) = x\alpha(x, y^2) + y\beta(x, y^2)$  for some  $\alpha, \beta \in \mathcal{E}_{u, v}$  with  $\alpha(0) > 0$ . By putting  $A = \sqrt{\alpha}, B = \beta(u, v) - \beta(0, v) + a(v)$ , we have (3.3). Define  $H_1, K, H_2$  by the form in Lemma 3.2. Then we have the commutativity of the diagram (3.1) and from (3.3) we see that  $\tilde{f} = k \circ f \circ H_2^{-1}$  satisfies (3.2). ■

Therefore the proof of Theorem A is reduced to show, for any  $f(x, y) = x + y + \text{higher term}$ , the existence of  $a \in \mathcal{E}_1$  in Proposition 3.4. To do this firstly we shall consider under the formal category and next we shall consider under  $C^\infty$  category to kill the flat terms.

**LEMMA 3.3.5.** *Any germ of type (III, III) with the form in Lemma 3.3 is equivalent to*

$$(\mathbb{R}, 0) \xleftarrow{x+y} (\mathbb{R}^2, 0) \xrightarrow{\gamma_1} (\mathbb{R}^2, 0) \xleftarrow{\gamma_2} (\mathbb{R}^2, 0) \xrightarrow{\tilde{f}} (\mathbb{R}, 0),$$

where  $\tilde{f}(x, 0) = x + \text{flat term}$ ,  $\tilde{f}(0, y) = y + \text{flat term}$ .

**Proof.** Define the map  $j^\infty : \mathcal{E}_1 \rightarrow \mathcal{E}_1/\mathcal{M}_1^\infty \cong \mathbb{R}[[t]]$  by  $j^\infty(a(t)) = \text{the Talyor expansion of } a(t)$ . By direct calculations we see that for any  $\bar{b}(t) \in \mathbb{R}[[t]]$  with  $\bar{b}(0) = 1$  there exists  $\bar{a}(t) \in \mathbb{R}[[t]]$  satisfying

(3.6)

$$\bar{a}(t^4) = \bar{b}^2(t)\bar{a}^4(t\bar{b}(t)), \quad \bar{a}(0) > 0.$$

For any  $f = x + y + \text{higher term}$ , we put  $\bar{b} = j^\infty(b)$ , where  $f(0, t)^{-1} \circ f(t, 0) = tb(t)$ . Then let  $\bar{a} \in \mathbb{R}[[t]]$  be a solution of (3.6). Applying the Borel's theorem there exists  $a \in \mathcal{E}_1$  such that  $j^\infty(a) = \bar{a}$ . Then we put  $h(t) = ta(t^2)$  and moreover we take a representative element  $k$  in  $t\sqrt{a(t^4)} \circ f(t, 0)^{-1} + \mathcal{M}_1^\infty$ . We define a  $(\gamma_1, \gamma_2)$ -compatible diffeomorphism germ by the same way as in the proof of Proposition 3.4. Then by using these coordinate changes we have the required result. ■

To kill the flat terms of  $\tilde{f}$  we shall apply Proposition 3.4. That is, since  $\tilde{f}(0, t)^{-1} \circ \tilde{f}(t, 0) = t(1 + \text{flat function})$ , in order to prove Theorem A it is sufficient to show

PROPOSITION 3.3.6. *For any  $b(t) \in \mathcal{E}_1$  of the form  $b(t) = 1 + \text{flat function}$  there exists  $a \in \mathcal{E}_1$  with  $a(0) > 0$  satisfying (3.5).*

To prove this we need the following Lemma. Let  $\theta : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$  be the inverse function germ of  $t \mapsto tb(t)$ , where  $b(t) = 1 + \text{flat function}$ . We see that  $\theta(x) = xc(x)$  where  $c(x) = 1 + \text{flat function}$ . Then from the equation (3.5), we see that  $a(\theta(x)^4) = (1/c(x))^2 a(x)^4$ . Thus we have

(3.7)

$$a(x) = c(x)^{\frac{1}{2}} a(\theta(x)^4)^{\frac{1}{4}}.$$

Then by direct calculations inductively we have

LEMMA 3.3.7. (i) *In the above equation (3.7), for any non-negative integer  $n$ ,  $a(x)$  satisfies*

$$a(x) = \left( \prod_{k=0}^n c(\sigma^k(x))^{1/4^k} \right)^{1/2} a(\sigma^{n+1}(x))^{1/4^{n+1}},$$

where  $\sigma(x) = \theta(x)^4$ ,  $\sigma^0(x) = x$ ,  $\sigma^n = \underbrace{\sigma \circ \dots \circ \sigma}_n$  ( $n \geq 1$ ).

(ii)  $\lim_{n \rightarrow \infty} a(\sigma^n(x))^{1/4^n} = 1$  on  $(\mathbb{R}, 0)$ .

Hence if  $a(x) \in \mathcal{E}_1$  satisfying the equation (3.5) exists, then  $a(x)$  must be  $a(x) = \left( \prod_{n=0}^{\infty} c(\sigma^n(x))^{1/4^n} \right)^{1/2}$ . Actually the following holds:

PROPOSITION 3.3.8.  $\left( \prod_{n=0}^{\infty} c(\sigma^n(x))^{1/4^n} \right)^{1/2}$  *exists and is  $C^\infty$ .*

Therefore proof of Proposition 3.6 is reduced to proof of Proposition 3.8. To prove Proposition 3.8 we need the following Lemmas.

We recall that  $(\prod_{n=0}^{\infty} f_n) = \lim_{N \rightarrow \infty} f_0 \cdots f_N$  exists and is  $C^\infty$  if and only if  $\lim_{N \rightarrow \infty} \sum_{k=0}^N \log f_k$  exists and is  $C^\infty$ , where  $f_k \in \mathcal{E}_1$  with  $f_k > 0$ .

Put  $S_N(x) = \sum_{n=0}^N \log c(\sigma^n(x))^{1/4^n} = \sum_{n=0}^N \frac{1}{4^n} \log c(\sigma^n(x))$ .

LEMMA 3.3.9.  $\left( \prod_{n=0}^{\infty} c(\sigma^n(x))^{1/4^n} \right)^{1/2}$  *is uniformly convergent.*

**Proof.** There is  $\varepsilon > 0$  such that  $|\sigma^n(x)| < |x|^{3^n} < \varepsilon < 1$ , which leads to  $|\frac{1}{4^n} \log c(\sigma^n(x))| < \frac{1}{4^n}$  for any  $n$  on  $|x| < \varepsilon$ . Hence  $\lim_{N \rightarrow \infty} S_N(x)$  is uniformly convergent on  $|x| < \varepsilon$ . ■

Next we shall show that  $S(x) = \lim_{N \rightarrow \infty} S_N(x)$  is of class  $C^\infty$ . Firstly we see that  $S(x) = \frac{1}{4}S(\sigma(x)) + \log c(x)$ . Actually from the definition of  $S_N$  we have

$$\frac{1}{4}S_N(\sigma(x)) = \sum_{n=0}^N \frac{1}{4^{n+1}} \log c(\sigma^{n+1}(x)) = S_{N+1}(x) - S_0(x).$$

Then by  $N \rightarrow \infty$  we have the required relation.

Therefore if  $S(x)$  is of class  $C^p$  ( $p = 0, 1, 2, \dots$ ), we inductively have

$$S^{(p)}(x) = \frac{1}{4}S^{(p)}(\sigma(x))\sigma'(x)^p + \lambda_p(x),$$

where  $S^{(p)}(x)$  is the  $p$ -th order differential of  $S(x)$ ,  $\lambda_p(x) \in \mathcal{E}_1$  depend on  $p$ . We put  $T(x) := S^{(p)}(x)$ ,  $\alpha(x) := \sigma'(x)^p$ ,  $\lambda(x) := \lambda_p(x)$ . Then we inductively have

$$\begin{aligned} T(x) &= \frac{1}{4^n} \alpha(x) \alpha(\sigma(x)) \alpha(\sigma^2(x)) \cdots \alpha(\sigma^{n-1}(x)) T(\sigma^n(x)) \\ &\quad + \sum_{i=0}^{n-1} \frac{1}{4^i} \alpha(x) \alpha(\sigma(x)) \cdots \alpha(\sigma^{i-1}(x)) \lambda(\sigma^i(x)). \end{aligned}$$

LEMMA 3.3.10. *There exists  $\varepsilon > 0$  such that*

$$\lim_{n \rightarrow \infty} \frac{1}{4^n} \alpha(x) \alpha(\sigma(x)) \alpha(\sigma^2(x)) \cdots \alpha(\sigma^{n-1}(x)) T(\sigma^n(x)) = 0$$

on  $|x| < \varepsilon$  for any  $p$ .

**Proof.** Clearly there exists  $\varepsilon \in (0, 1)$  such that  $|\sigma'(x)| < |x|^2$  on  $|x| < \varepsilon$ . Thus we see that

$$\left| \frac{1}{4^n} \sigma'(x)^p \sigma'(\sigma(x))^p \cdots \sigma'(\sigma^{n-1}(x))^p T(\sigma^n(x)) \right| < \frac{1}{4^n} |x|^{2pn} |T(\sigma^n(x))|$$

on  $|x| < \varepsilon$  for any  $p$ . Then the right hand side is convergent to 0 on  $|x| < \varepsilon$  if  $n \rightarrow \infty$ .  $\blacksquare$

LEMMA 3.3.11. *There exists  $\varepsilon > 0$  such that for any  $\alpha(x) \in \mathcal{E}_1$  with  $\alpha(0) = \alpha'(0) = 0$  and for any  $\lambda(x)$  which is  $C^1$  class function germ on  $(\mathbb{R}, 0)$  with  $\lambda(0) = \lambda'(0) = 0$  the followings hold:*

Let  $u_n(x) = \alpha(x) \alpha(\sigma(x)) \alpha(\sigma^2(x)) \cdots \alpha(\sigma^n(x)) \lambda(\sigma^{n+1}(x))$ .

(i)  $\lim_{N \rightarrow \infty} \sum_{n=0}^N u_n(x)$  is pointwise convergent on  $|x| < \varepsilon$ .

(ii)  $\lim_{N \rightarrow \infty} \sum_{n=0}^N u_n(x)$  is of class  $C^1$ .



**Proof of (i).** We can take an  $\varepsilon_0 \in (0, \frac{1}{2})$  such that for any  $\alpha, \lambda$  in this Lemma there exists a positive number  $M_{\alpha,\lambda} \in \mathbb{R}$  such that the followings hold for any  $|x| < \varepsilon_0$ :

$$|\alpha(x)| < M_{\alpha,\lambda}, |\alpha'(x)| < M_{\alpha,\lambda}, |\lambda(x)| \leq M_{\alpha,\lambda}|x|, |\lambda'(x)| \leq M_{\alpha,\lambda}|x|.$$

Take  $\varepsilon_\sigma \in (0, \frac{1}{2})$  such that  $|\sigma| < |x|^3$  (hence  $|\sigma^n| < |x|^{3^n}$  for any  $n$ ) on  $|x| < \varepsilon_\sigma$ . Let  $\varepsilon_1 = \min\{\varepsilon_0, \varepsilon_\sigma\}$ . Then we see  $|u_n| < M_{\alpha,\lambda}^{n+2}|\sigma^{n+1}(x)| < M_{\alpha,\lambda}^{n+2}|x|^{3^{n+1}}$  on  $|x| < \varepsilon_1$ . We easily verify that the series  $\sum M_{\alpha,\lambda}^{n+2}|x|^{3^{n+1}}$  converges pointwise on  $|x| < \varepsilon_1$  and hence  $\sum u_n$  is also. ■

**Proof of (ii).** Since  $u_n(x)$  is of class  $C^1$ , it is sufficient to show that there exists  $\varepsilon > 0$  independent of  $\lambda$  such that the series  $\sum u'_n(x)$  converges uniformly on  $|x| < \varepsilon$ . Take  $\varepsilon_2 > 0$  such that  $|(\sigma^i)'| < 1$  for any  $i$  on  $|x| < \varepsilon_2$ . Let  $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ , where  $\varepsilon_1$  is defined just as before. Then we have  $|u'_n(x)| < (n+2)M_{\alpha,\lambda}^{n+2}|x|^{3^{n+1}}$  on  $|x| < \varepsilon$ . Hence we easily verify that  $\sum u'_n(x)$  converges pointwise.

Moreover we have  $|u'_n(x)| < (n+2)M_{\alpha,\lambda}^{n+2}(1/2^{3^{n+1}}) =: K_n$  on  $|x| < \varepsilon$ . Since  $\sqrt[n]{K_n} \rightarrow 0$ ,  $\sum K_n$  converges. Therefore  $\sum u'_n(x)$  converges uniformly on  $|x| < \varepsilon$ . ■

Now we are ready to show that  $S(x)$  is of class  $C^\infty$ .

**Proof of Proposition 3.8.** We recall the relation (3.8) of  $S(x)$ . In (3.8), if  $S(x)$  is of class  $C^p$ , by direct calculations we see  $\lambda_0 = \log c, \lambda_1 = \frac{c'}{c}, \lambda_p = \frac{1}{4}S^{(p-1)}(\sigma) \cdot (p-1)(\sigma')^{p-2}\sigma'' + \lambda'_{p-1}$  =sums and products of  $\lambda_1^{(p-1)}, S'(\sigma), S''(\sigma), \dots, S^{(p-1)}(\sigma), \sigma, \sigma', \sigma'', \dots, \sigma^{(p)}$  for  $p \geq 2$ . Since  $c, \sigma$  are of class  $C^\infty$ , if  $S(x)$  is of class  $C^p$ , then  $\lambda_p$  is of class  $C^1$ . Then by putting  $\lambda = \lambda_p, \alpha = (\sigma')^p$ , from the relation (3.9), Lemma 3.10 and Lemma 3.11 we can inductively prove that  $S^{(p)} = \sum_{n=0}^{\infty} \frac{1}{4^{n+1}}u_n + \lambda_p$  is of class  $C^1$  for any  $p$ . ■

#### 4. TOPOLOGICAL RIGIDITY FOR TYPE (III, III)

In this section we shall prove the following which is a part of Theorem B:

PROPOSITION 4.4.1. *If two diagrams of type (III, III) are topologically equivalent, then they are  $C^\infty$ -equivalent.*

One of the essential tool to prove theorem B is the following Theorem of rigidity of webs. Let  $\mathcal{W} = (\mathcal{F}_1, \dots, \mathcal{F}_d)$  be a configuration of  $d$  foliations  $\mathcal{F}_j, j = 1, \dots, d (d \geq 2)$  in a domain  $U$  of  $\mathbb{R}^2$  with codimension 1. Define the set  $S(\mathcal{W})$  by  $S(\mathcal{W}) :=$  the set of points at which  $\mathcal{F}_i$  and  $\mathcal{F}_j$  ( $1 \leq i < j \leq d$ ) are non-transversal. We call  $\mathcal{W}$  a *non-singular  $d$ -web* or briefly  *$d$ -web* (resp. *singular  $d$ -web*) on  $U$  of codim 1 if  $S(\mathcal{W}) = \emptyset$  (resp.  $S(\mathcal{W}) \neq \emptyset$ ) and call  $S(\mathcal{W})$  the *singular set* of  $\mathcal{W}$ . Therefore locally a (singular)  $d$ -web is a configuration  $\mathcal{W} = (f_1, \dots, f_d)$  of  $d$  submersion germs  $f_j : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$ . Then the singular set  $S(\mathcal{W})$  is given by  $S(\mathcal{W}) = \{p \in (\mathbb{R}^2, 0) \mid df_i \wedge df_j(p) = 0, 1 \leq i < j \leq d\}$ .

THEOREM 4.4.2. (*Rigidity of webs[4]*) *Let  $\mathcal{W} = (f_1, f_2, f_3), \mathcal{W}' = (f'_1, f'_2, f'_3)$  be two 3-webs on  $(\mathbb{R}^2, 0)$ . If there exist homeomorphism germs  $\Phi : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0), h_j : (\mathbb{R}, 0) \rightarrow$*

$(\mathbb{R}, 0)$  such that  $h_j \circ f_j = f'_j \circ \Phi$  for any  $j = 1, 2, 3$ , then  $\Phi, h_j (j = 1, 2, 3)$  are  $C^\infty$ -diffeomorphism germs.

As a corollary of Theorem 4.2 we see that rigidity of webs holds for every  $d$ -webs with  $d \geq 3$ .

Suppose that two diagrams of type (III, III) are topologically equivalent. That is, from Theorem A, we suppose that the following diagram commute:

(4.1)

$$\begin{array}{ccccccccc} (\mathbb{R}, 0) & \xleftarrow{x+y} & (\mathbb{R}^2, 0) & \xrightarrow{\gamma_1} & (\mathbb{R}^2, 0) & \xleftarrow{\gamma_2} & (\mathbb{R}^2, 0) & \xrightarrow{f} & (\mathbb{R}, 0) \\ h_1 \downarrow & & H_1 \downarrow & & K \downarrow & & \downarrow H_2 & & \downarrow h_2 \\ (\mathbb{R}, 0) & \xleftarrow{X+Y} & (\mathbb{R}^2, 0) & \xrightarrow{\gamma_1} & (\mathbb{R}^2, 0) & \xleftarrow{\gamma_2} & (\mathbb{R}^2, 0) & \xrightarrow{\tilde{f}} & (\mathbb{R}, 0) \end{array}$$

where  $f, \tilde{f} \in \mathcal{E}_2$  with  $f(x, 0) = f(0, x) = x$ ,  $\tilde{f}(\tilde{x}, 0) = \tilde{f}(0, \tilde{x}) = \tilde{x}$  and  $h_i, H_i (i = 1, 2), K$  are homeomorphism germs. We put  $H_1 = (X, Y)$ ,  $H_2 = (\tilde{X}, \tilde{Y})$ ,  $K = (U, V)$ .

We remark that there exists a 4-web in the domain  $\{(u, v) \in (\mathbb{R}^2, 0) | u > 0 \text{ and } v > 0\}$  which are constructed by  $\gamma_i(f_i^{-1}(t)), i = 1, 2$ . Therefore from Theorem 4.2 we see that  $K$  restricted to  $\{(u, v) \in (\mathbb{R}^2, 0) | u > 0 \text{ and } v > 0\}$  is a  $C^\infty$ -diffeomorphism germ.

LEMMA 4.4.3. *We have the followings:*

- (i)  $h_1, h_2$  are  $C^\infty$ -diffeomorphism germs and moreover  $h_1 = h_2 = id_{\mathbb{R}}$
- (ii)  $H_1 = id_{\mathbb{R}^2}$
- (iii)  $H_2$  is a  $C^\infty$ -diffeomorphism germ, moreover  $H_2$  restricted to  $\{(x, y) \in (\mathbb{R}^2, 0) | y \geq 0 \text{ or } x = 0\}$  is the identity map germ.

**Proof of (i).** We take a point  $p \in \{x + y = 0\} \cap \{x > 0, y < 0\}$  such that  $p$  is sufficiently near 0. Then we consider a non-singular  $C^\infty$ -curve  $T$  at  $p$  such that  $T$  and each lines  $x + y = t (t \in (\mathbb{R}, 0))$  are transversal. Then  $x + y$  restricted to  $T$  is a  $C^\infty$ -diffeomorphism germ. Since  $\gamma_1|_{\{x > 0, y < 0\}}$  and  $K|_{\{u > 0, v > 0\}}$  are  $C^\infty$ -diffeomorphism germs, from (4.1) we see that  $H_1|_{\{x > 0, y < 0\}}$  is a  $C^\infty$ -diffeomorphism germ. From (4.1) we have  $h_1 \circ (x + y)|_T = (X + Y) \circ H_1|_T$ , namely  $h_1$  is a  $C^\infty$ -diffeomorphism germ. For  $h_2$ , similarly take a point  $p \in f^{-1}(0) \cap \{x < 0, y > 0\}$  which is sufficiently near 0 and then consider a curve  $T$  at  $p$  such that  $T$  and each fibres of  $f$  are transversal. Then by the similar argument we see that  $h_2$  is a  $C^\infty$ -diffeomorphism germ.

Hence there exist  $A, B \in \mathcal{E}_{u,v}$  such that  $h_1(x + y) = A(x, y^2) + yB(x, y^2)$ . Similarly there exist  $\tilde{A}, \tilde{B} \in \mathcal{E}_{u,v}$  such that  $h_2 \circ f(x, y) = \tilde{A}(x^2, y) + x\tilde{B}(x^2, y)$ . We remark that in (4.1)  $H_1, K, H_2$  preserve both the horizontal-axis and the vertical-axis in each spaces. From (4.1) we have  $U(x, y^2) = A(x, y^2)$ ,  $V(x, y^2) = y^2B(x, y^2)^2$  and similarly  $U(x^2, 0) = x^2\tilde{B}(x^2, 0)^2$ ,  $V(0, y) = \tilde{A}(0, y)$ . Thus we have  $h_1(y) = yB(0, y^2)$  and  $h_1(x) = x\tilde{B}(x, 0)^2$  for  $x \geq 0$ . Similarly  $h_2(x) = x\tilde{B}(x^2, 0)$  and  $h_2(y) = yB(0, y^2)^2$  for  $y \geq 0$ . This means that  $h_1, h_2$  are increasing and odd. Therefore there exist  $a_i \in \mathcal{E}_1$  with  $a_i(0) > 0$  such that

$h_i(t) = ta_i(t^2)$ ,  $i = 1, 2$ . Hence  $a_i(t)^4 = a_i(t^4)$  for  $t \geq 0$ ,  $i = 1, 2$ . Thus we see  $a_i(t) \equiv 1$  for  $t \geq 0$ , namely  $h_i(t) = t$  on  $(\mathbb{R}, 0)$ . ■

**Proof of (ii).** From (i) it is clear. ■

**Proof of (iii).** For  $f = x + y + \text{higher term} \in \mathcal{E}_2$  consider the map germ  $J_f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$  defined by  $J_f(x, y) = (f(x, y), f(-x, y))$ . Clearly  $J_f$  is a  $C^\infty$ -diffeomorphism germ. Since  $H_2(-x, y) = (-\tilde{X}(x, y), \tilde{Y}(x, y))$  and  $h_2 = id$ , we have  $J_f = J_{\tilde{f}} \circ H_2$ . Namely  $H_2$  is a  $C^\infty$ -diffeomorphism germ. The latter part of (iii) is clear by (i) and (ii). ■

From Lemma 4.3 (iii) and the  $\gamma_2$ -compatibility, we can put  $H_2 = (xA(x^2, y), yB(x^2, y))$  for some  $A, B \in \mathcal{E}_{u,v}$  with  $A = 1$  on  $\{u \geq 0 \text{ and } v \geq 0\}$ ,  $B = 1$  on  $\{u \geq 0 \text{ and } v \geq 0\} \cup \{u = 0\}$ . Then we see that  $K = (uA^2, vB)$  on  $\{u \geq 0\}$ . On the other hand  $K(u, v) = (u, v)$  on  $\{v \geq 0\}$ . Now we need the following Proposition.

PROPOSITION 4.4.4. *Let  $h \in \mathcal{E}_{u,v}$  with  $h = 1$  on  $\{u \geq 0 \text{ and } v \geq 0\}$ . Then there exists  $\tilde{h} \in \mathcal{E}_{u,v}$  such that  $\tilde{h} = h$  on  $\{u \geq 0\}$  and  $\tilde{h} = 1$  on  $\{u \leq 0 \text{ and } v \geq 0\}$ .*

If this Proposition 4.4 is proved, the proof of Proposition 4.1 is finished by the following.

Proof of Proposition 4.1. For the above  $A, B \in \mathcal{E}_{u,v}$  we take its extension  $\tilde{A}, \tilde{B}$  with the same property in Proposition 4.4. Then define the  $C^\infty$ -diffeomorphism germ  $\tilde{K} : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$  by  $\tilde{K} = (u\tilde{A}(u, v)^2, v\tilde{B}(u, v))$ . Then we see that  $\tilde{K} \circ \gamma_1 = \gamma_1$  and  $\tilde{K} \circ \gamma_2 = \gamma_2 \circ H_2$ . This completes the proof. ■

In order to prove Proposition 4.4 we apply the theory of *regularly situated sets* by Lojasiewicz as follows (See [11], [12]p.12): Let  $\Omega$  be an open set in  $\mathbb{R}^n$  and let  $X$  be a closed subset of  $\Omega$ . Let  $\mathcal{E}(X)$  be the ring of all Whitney functions of infinite order on  $X$ .

Let  $X, Y$  be closed subsets of  $\Omega$ . Let  $\delta : \mathcal{E}(X \cup Y) \rightarrow \mathcal{E}(X) \oplus \mathcal{E}(Y)$  be the diagonal mapping defined by

$$\delta(F) = (F|_X, F|_Y).$$

Let  $\pi : \mathcal{E}(X) \oplus \mathcal{E}(Y) \rightarrow \mathcal{E}(X \cap Y)$  be the mapping defined by

$$\pi(F, G) = F|_X - G|_Y.$$

DEFINITION 4.4.5. ([11, 12]) *Two closed subsets  $X, Y$  of an open set  $\Omega$  are said to be regularly situated if the sequence*

$$0 \longrightarrow \mathcal{E}(X \cup Y) \xrightarrow{\delta} \mathcal{E}(X) \oplus \mathcal{E}(Y) \xrightarrow{\pi} \mathcal{E}(X \cap Y) \longrightarrow 0$$

*is exact.*

THEOREM 4.4.6. (Lojasiewicz) *Given  $X, Y$  closed in an open set  $\Omega$  a necessary and sufficient condition that they are regularly situated is the following: Either  $X \cap Y = \emptyset$  or*

( $\Lambda$ ) Given any pair of compact sets  $K \subset X, L \subset Y$ , there exists a pair of constants  $C > 0$  and  $\alpha > 0$  such that, for every  $x \in K$ , one has

$$d(x, L) \geq C d(x, X \cap Y)^\alpha,$$

where  $d$  denoting the euclidean distance in  $\mathbb{R}^n$ .

We apply Theorem of Lojasiewicz for the following particular case. Consider closed subsets  $X, Y$  of  $\mathbb{R}^2$  defined by

$$X = \{re^{i\theta} | 0 \leq r \leq \varepsilon, -\frac{\pi}{4} \leq \theta \leq \frac{2}{3}\pi\}, Y = \{re^{i\theta} | 0 \leq r \leq \varepsilon, \frac{5}{6}\pi \leq \theta \leq \frac{7}{4}\pi\},$$

where  $\varepsilon$  is a fixed positive number.

LEMMA 4.4.7.  $X, Y$  are regularly situated.

**Proof.** We can easily see  $d(p, Y) = d(p, X \cap Y)$  for every  $p \in \{re^{i\theta} | 0 \leq r \leq \varepsilon, -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{3}\}$  and also see that  $d(p, Y) = d(p, 0) \sin(\frac{5}{6}\pi - \arg p)$ ,  $d(p, X \cap Y) = d(p, 0)$  for every  $p \in \{re^{i\theta} | 0 \leq r \leq \varepsilon, \frac{\pi}{3} \leq \theta \leq \frac{2}{3}\pi\}$ . Thus we have  $d(p, Y) \geq \frac{1}{2}d(p, X \cap Y)$  for any  $p \in X$ . Then for  $X, Y$  obviously the condition ( $\Lambda$ ) holds. ■

Clearly to prove Proposition 4.4 it is enough to show the following.

LEMMA 4.4.8. Let  $h : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$  be a  $C^\infty$ -function germ such that  $h = 0$  on  $\{(x, y) | x \geq 0 \text{ and } y \leq 0\}$ . Then there exists a  $C^\infty$ -function germ  $\tilde{h} : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$  such that  $\tilde{h} = h$  on  $\{x \geq 0\}$  and  $\tilde{h} = 0$  on  $\{y \leq 0\}$ .

**Proof.** We take a sufficiently small  $\varepsilon > 0$ . Then we can define  $f \in \mathcal{E}(X)$  by  $h|_X$  and define  $g \in \mathcal{E}(Y)$  by  $0|_Y$ . Obviously  $(f, g) \in \text{Ker } \pi$ . Then by Lemma 4.7 there exists  $F \in \mathcal{E}(X \cup Y)$  such that  $\delta(F) = (f, g)$ . By the definition of Whitney function there exists a  $C^\infty$ -function  $\tilde{F}$  defined in a neighbourhood of  $X \cup Y$  in  $\mathbb{R}^2$  such that  $j^\infty \tilde{F}(p) = F(p)$  in  $X \cup Y$ . Put  $\tilde{h} :=$ the germ of  $\tilde{F}$  at 0, which is the required  $C^\infty$ -function germ. ■

## 5. RIGIDITY FOR EACH TYPES EXCEPT (III, III)

In this section we shall complete proof of Theorem B. Firstly we recall the following normal forms which have been given in [3, 10, 14].

THEOREM 5.5.1. The normal forms of each generic types except (III, III) for the diagrams  $(\mathbb{R}, 0) \xleftarrow{f_1} (\mathbb{R}^2, 0) \xrightarrow{\gamma_1} (\mathbb{R}^2, 0) \xleftarrow{\gamma_2} (\mathbb{R}^2, 0) \xrightarrow{f_2} (\mathbb{R}, 0)$  are the followings:

$$(I, I)^0 \quad \begin{aligned} f_1 &= y_1, & \gamma_1 &= (x_1, y_1); \\ f_2 &= x_2, & \gamma_2 &= (x_2, y_2). \end{aligned}$$

- $(I, I)^1$   $f_1 = y_1, \gamma_1 = (x_1, y_1);$   
 $f_2 = x_2^2 + y_2, \gamma_2 = (x_2, y_2).$
- $(I, I)^2$   $f_1 = y_1, \gamma_1 = (x_1, y_1);$   
 $f_2 = x_2^3 + x_2 y_2 + y_2, \gamma_2 = (x_2, y_2).$
- $(II, I)$   $f_1 = x_1^2 \pm y_1^2, \gamma_1 = (x_1, y_1);$   
 $f_2 = x_2, \gamma_2 = (x_2, y_2).$
- $(III, I)^0$   $f_1 = x_1 + y_1, \gamma_1 = (x_1, y_1^2);$   
 $f_2 = x_2 + \theta(x_2, y_2), \gamma_2 = (x_2, y_2),$   
*where  $\theta \in \mathcal{M}_{x_2, y_2}$  with  $\theta(x_2, 0) = 0$ .*
- $(III, I)^1$   $f_1 = x_1 + y_1, \gamma_1 = (x_1, y_1^2);$   
 $f_2 = x_2^2 + \theta(x_2, y_2), \gamma_2 = (x_2, y_2),$   
*where  $\theta \in \mathcal{M}_{x_2, y_2}$  with  $\theta(x_2, 0) = 0, \frac{\partial \theta}{\partial y_2}(0) \neq 0$ .*
- $(IV, I)$   $f_1 = x_1^2 + y_1, \gamma_1 = (x_1, y_1^2);$   
 $f_2 = x_2 + \theta(x_2, y_2), \gamma_2 = (x_2, y_2),$   
*where  $\theta \in \mathcal{M}_{x_2, y_2}$  with  $\theta(x_2, 0) = 0, \frac{\partial \theta}{\partial y_2}(0) \neq 0$ .*
- $(V, I)$   $f_1 = x_1 + x_1 y_1 + y_1^3, \gamma_1 = (x_1, y_1^2);$   
 $f_2 = x_2 + \theta(x_2, y_2), \gamma_2 = (x_2, y_2),$   
*where  $\theta \in \mathcal{M}_{x_2, y_2}$  with  $\theta(x_2, 0) = 0, \frac{\partial \theta}{\partial y_2}(0) \neq 0, \frac{\partial^2 \theta}{\partial y_2^2}(0) \neq 3$ .*
- $(VI, I)$   $f_1 = y_1 + \alpha \circ \gamma_1, \gamma_1 = (x_1, y_1^3 + x_1 y_1);$   
 $f_2 = x_2 + \theta(x_2, y_2), \gamma_2 = (x_2, y_2),$   
*where  $\alpha \in \mathcal{M}_{u, v}, \theta \in \mathcal{M}_{x_2, y_2}$  with  $\theta(x_2, 0) = 0$ .*

For the normal forms with functional moduli except type  $(III, III)$ , that is, for the normal forms of  $(III, I)^0, (III, I)^1, (IV, I), (V, I)$  and  $(VI, I)$  we may skip  $\gamma_2$  to consider the topological rigidity problem because  $\gamma_2 = id$ .

### 5.1. Proof of rigidity for $(III, I)^0, (IV, I), (V, I)$

We uniformly prove for these types. In this proof we always suppose that  $\gamma = (x, y^2) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ . Assume that two normal forms of the same type (i.e.  $(III, I)^0, (IV, I), (V, I)$ ) are topologically equivalent:

(5.1)

$$\begin{array}{ccccccc}
 (\mathbb{R}, 0) & \xleftarrow{g} & (\mathbb{R}^2, 0) & \xrightarrow{\gamma} & (\mathbb{R}^2, 0) & \xrightarrow{f} & (\mathbb{R}, 0) \\
 h \downarrow & & H \downarrow & & K \downarrow & & \downarrow k \\
 (\mathbb{R}, 0) & \xleftarrow{g} & (\mathbb{R}^2, 0) & \xrightarrow{\gamma} & (\mathbb{R}^2, 0) & \xrightarrow{\tilde{f}} & (\mathbb{R}, 0)
 \end{array}$$

where  $h, H, K, k$  are homeomorphism germs.

**Step 1.**  $h, k$  are  $C^\infty$ -diffeomorphism germs.

*Case  $(III, I)^0$ .* The diagrams of this type have a 3-web structure on  $\{v > 0\}$ . By the rigidity of webs,  $K|_{\{v > 0\}}$  is a  $C^\infty$ -diffeomorphism germ. Since  $\gamma|_{\{v > 0\}}$  is a  $C^\infty$ -diffeomorphism

germ, from (5.1) we see that  $H|_{\{y>0\}}$  is a  $C^\infty$ -diffeomorphism germ. Take a point  $p \in g^{-1}(0) \cap \{y > 0\}$  such that  $p$  is sufficiently near 0. Then we can consider a non-singular curve  $T$  at  $p$  such that  $T$  and each fibres of  $g$  are transversal. Namely  $g|_T$  gives a  $C^\infty$ -diffeomorphism germ. Hence  $g$  restricted to  $H(T)$  is a  $C^\infty$ -diffeomorphism germ. Then from (5.1)  $h \circ g|_T = g|_{H(T)} \circ H|_T$ , namely  $h$  is a  $C^\infty$ -diffeomorphism germ.

Similarly we take a point  $q \in f^{-1}(0) \cap \{v > 0\}$  such that  $q$  is sufficiently near 0 and then consider a non-singular curve  $S$  at  $q$  such that  $S$  and each fibres of  $f$  are transversal. Then we see that  $k = f|_{K(S)} \circ K|_S \circ f|_S^{-1}$  is a  $C^\infty$ -diffeomorphism germ.

*Case (IV, I).* In this case there is a singular 3-web on the domain  $\{v > 0\}$  and its singular set is the  $v$ -axis ( $v > 0$ ). That is, there is a 3-web on the domain  $\{(u, v) | u \neq 0, v > 0\}$ . By the rigidity of webs,  $K$  restricted to  $\{u \neq 0, v > 0\}$  is a  $C^\infty$ -diffeomorphism germ. Hence we see that  $H$  restricted to  $\{x > 0, y < 0\}$  is a  $C^\infty$ -diffeomorphism germ. Take a point  $p \in g^{-1}(0) \cap \{x > 0, y < 0\}$  such that  $p$  is sufficiently near 0. Then taking a curve  $T$  at  $p$  with the same property as the case  $(III, I)^0$  we see that  $h$  is a  $C^\infty$ -diffeomorphism germ.

On the other hand by the generic condition, since  $f^{-1}(0)$  and  $v$ -axis ( $v > 0$ ) are transversal we can take a point  $q \in f^{-1}(0) \cap \{u \neq 0, v > 0\}$ . Then we can take a curve  $S$  at  $q$  in  $\{u < 0, v > 0\}$  or  $\{u > 0, v > 0\}$  which has the same property as in the case  $(III, I)^0$ . Hence  $k$  is a  $C^\infty$ -diffeomorphism germ.

*Case (V, I).* In this case there is a singular 3-web on  $\{v > 0\}$  such that 3 foliations are given by the level curves  $\gamma(g^{-1}(t))$  and  $f^{-1}(t)$ . That is, the singular 3-web  $\mathcal{W}$  is given by

$$f_1 = u + (u + v)\sqrt{v}, f_2 = u - (u + v)\sqrt{v}, f_3 = f = u + bv + \theta(u, v)$$

for  $v > 0$ , where  $b \neq 0, \theta \in \mathcal{M}_{u,v}^2$ . We decompose the singular set  $S(\mathcal{W})$  by  $S(\mathcal{W}) = S_{1,2} \cup S_{2,3} \cup S_{3,1}$ , where  $S_{i,j} = \{(u, v) | v > 0, \det J_{(f_i, f_j)}(u, v) = 0\} (i \neq j)$ ,  $J_{(f_i, f_j)}(u, v)$  denoting the Jacobi matrix of  $(f_i, f_j)$  at  $(u, v)$ . We have  $S_{1,2} = \{u + 3v = 0, v > 0\}$ . We see that the tangent directions of the level curves of  $f_1$  and  $f_2$  are vertical at every points in  $S_{1,2}$ .

To show the Step 1, we shall apply the same technique as the case  $(III, I)^0$ . So that we shall avoid the set  $S_{2,3} \cup S_{3,1}$  as follows. Let  $m \in \mathbb{R}$ . Then we have  $\det J_{(f_3, f_1)}(mv, v) = (1 + \frac{\partial \theta}{\partial u}(mv, v)) \frac{m+3}{2} \sqrt{v} - (b + \frac{\partial \theta}{\partial v}(mv, v))(1 + \sqrt{v})$ . Since  $\lim_{v \rightarrow 0} \det J_{(f_3, f_1)}(mv, v) = -b \neq 0$ , there exists  $\delta_1(m) > 0$  such that  $\det J_{(f_3, f_1)}(mv, v)$  does not vanish for  $\delta_1(m) > v > 0$ . We can choose  $\delta_1(m)$  such that  $\delta_1(m)$  is a continuous function with respect to  $m$ . Similarly there exists  $\delta_2(m) > 0$  such that  $\det J_{(f_3, f_2)}(mv, v) \neq 0$  for  $\delta_2(m) > v > 0$ . Now for  $f$  we take a sufficiently large  $l \in \mathbb{R}$  such that  $f^{-1}(0)$  and  $S_{1,2}$  lie in the set  $\{(mv, v) | -l < m < l, v > 0\}$ . Put  $\delta = \min\{\delta_1(m), \delta_2(m)\}$  for  $-l \leq m \leq l$  and consider the domain  $D_f = \{(mv, v) | -l < m < l, 0 < v < \delta\}$ . Then we see that there is a non-singular 3-web on  $D_f - S_{1,2}$ . By rigidity of webs we see that  $K$  restricted to  $D_f - S_{1,2}$  is a  $C^\infty$ -diffeomorphism germ. Therefore  $H$  restricted to  $\gamma^{-1}(D_f - S_{1,2}) \cap \{y > 0\}$  is a  $C^\infty$ -diffeomorphism germ. Take a point  $p \in g^{-1}(0) \cap \{y > 0\}$  such that  $p$  is sufficiently near 0. We can take a curve  $T$  at  $p$  in  $\gamma^{-1}(D_f - S_{1,2}) \cap \{y > 0\}$  such that  $T$  and each fibres of  $g$  around 0 are transversal. Then by the same argument as the case  $(III, I)^0$ , we see that  $h$  is a  $C^\infty$ -diffeomorphism germ.

Since  $f^{-1}(0)$  and  $S_{1,2}$  are transversal by the generic condition, we can take a point  $q \in f^{-1}(0) \cap (D_f - S_{1,2})$  and moreover we can take a curve  $S$  at  $q$  in  $D_f - S_{1,2}$  such that  $S$  and each fibres of  $f$  around 0 are transversal. Then by the same argument as the case  $(III, I)^0$ , we see that  $k$  is a  $C^\infty$ -diffeomorphism germ.

**Step 2.**  $H$  is a  $C^\infty$ -diffeomorphism germ.

For the normal forms  $(g, \gamma, f)$  of each cases we define a  $C^\infty$ -map germ  $\Phi_f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$  by  $\Phi_f = (g, f \circ \gamma)$ . Then we have  $h \times k \circ \Phi_f = \Phi_{\tilde{f}} \circ H$ .

*Cases  $(III, I)^0, (IV, I)$ .* we see that  $\Phi_f$  is a  $C^\infty$ -diffeomorphism germ. Hence from Step 1,  $H$  is a  $C^\infty$ -diffeomorphism germ.

*Case  $(V, I)$ .* From a generic condition “the tangent cone of  $\gamma(g^{-1}(0))$  and  $f^{-1}(0)$  are transversal” (i.e.  $\frac{\partial f}{\partial v}(0) \neq 0$ ), by direct calculations we see that  $\Phi_f$  is a fold map germ. Thus by Step 1 we see that  $H$  is a  $C^\infty$ -diffeomorphism germ because if  $\gamma \circ H_1 = H_2 \circ \gamma$ , where  $H_1$  is a homeomorphism germ and  $H_2 = (U, V)$  is a  $C^\infty$ -diffeomorphism germ, then we have  $H_1 = (U(x, y^2), y\sqrt{V_1(x, y^2)})$  for some  $V_1 \in \mathcal{E}_2$  with  $V_1(0) \neq 0$ .

**Step 3.** A  $C^\infty$ -extension of  $K$  restricted to the web domain.

We shall construct a  $C^\infty$ -diffeomorphism germ  $\tilde{K}$  preserving the commutativity (5.1) even if we replace  $K$  with  $\tilde{K}$  as follows: By the coordinate change  $(f, v)$  in the source of  $f$ , we may suppose  $f = \pi_1$ , where  $\pi_1$  is the first projection  $\pi_1(u, v) = u$ . Similarly  $\tilde{f} = \pi_1$ . Hence it is enough to construct a  $C^\infty$ -diffeomorphism germ  $\hat{K} : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$  such that the following diagram commutes

$$\begin{array}{ccccc} (\mathbb{R}^2, 0) & \xrightarrow{\delta} & (\mathbb{R}^2, 0) & \xrightarrow{\pi_1} & (\mathbb{R}, 0) \\ H \downarrow & & \hat{K} \downarrow & & \downarrow k \\ (\mathbb{R}^2, 0) & \xrightarrow{\tilde{\delta}} & (\mathbb{R}^2, 0) & \xrightarrow{\pi_1} & (\mathbb{R}, 0) \end{array}$$

where  $\delta = (f, v) \circ \gamma, \tilde{\delta} = (\tilde{f}, v) \circ \gamma$ . Let  $H = (X, Y)$ . Then from Step 2 and by the Malgrange preparation theorem there exists  $A, B \in \mathcal{E}_{u,v}$  such that  $Y = A \circ \delta + vB \circ \delta$ . From (5.1) we see that  $A \circ \delta = 0$  in  $(\mathbb{R}^2, 0)$ . Then put  $\hat{K}(u, v) = (k(u), vB(u, v)^2)$ . From (5.1) and Steps 1, 2 we see that  $\hat{K}$  is the required  $C^\infty$ -diffeomorphism germ.

This completes the proof of rigidities for  $(III, I)^0, (IV, I), (V, I)$ . ■

### 5.2. Proof of rigidity for $(III, I)^1$

The following was shown in [4].

LEMMA 5.5.2. *Let  $(\mathbb{R}, 0) \xleftarrow{f} (\mathbb{R}^2, 0) \xrightarrow{\gamma} (\mathbb{R}^2, 0)$  be of type  $(III)$ . Then only using  $C^\infty$ -coordinate changes in the source and the target of  $\gamma$ , without any coordinate changes in the target of  $f$ , the normal form of  $(III)$  is given by  $f = x + y, \gamma = (x, y^2)$ .*

We suppose that  $\gamma = (x, y^2) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ . From Theorem 5.1 the normal form of  $(III, I)^1$  is  $(x+y, \gamma, f)$ ,  $f(u, v) = u^2$ ,  $\frac{\partial f}{\partial v}(0) \neq 0$ . Assume that two normal forms of  $(III, I)^1$  are topologically equivalent:

(5.2)

$$\begin{array}{ccccccc} (\mathbb{R}, 0) & \xleftarrow{x+y} & (\mathbb{R}^2, 0) & \xrightarrow{\gamma} & (\mathbb{R}^2, 0) & \xrightarrow{f} & (\mathbb{R}, 0) \\ h \downarrow & & H \downarrow & & K \downarrow & & \downarrow k \\ (\mathbb{R}, 0) & \xleftarrow{X+Y} & (\mathbb{R}^2, 0) & \xrightarrow{\gamma} & (\mathbb{R}^2, 0) & \xrightarrow{\tilde{f}} & (\mathbb{R}, 0) \end{array}$$

where  $h, H, K, k$  are homeomorphism germs. We remark that if  $f(u, v) = -v + u^2$ , then the parabola  $f = 0$  is the singular set of the singular 3-web on  $\{v > 0\}$  associated with this normal form. However clearly the type  $(III, I)^1$  with  $f(u, v) = -v + u^2$  is topologically equivalent to only oneself. Hence we may suppose that  $f(u, v) \neq -v + u^2$ . Then there exists a 3-web structure on  $\{v > 0\}$ . We remark that from (5.2)  $\frac{\partial f}{\partial v}(0) < 0$  (resp.  $> 0$ ) if and only if  $\frac{\partial \tilde{f}}{\partial v}(0) < 0$  (resp.  $> 0$ ) because the level curves  $f^{-1}(0)$  with  $\frac{\partial f}{\partial v}(0) > 0$  do not lie in the web domain  $\{v > 0\}$ . We see that  $K|_{\{v > 0\}}$  and  $h$  are  $C^\infty$ -diffeomorphism germs by the same argument as in  $(III, I)^0$ .

By Lemma 5.2 we have the following  $C^\infty$ -commutative diagram

$$\begin{array}{ccccccc} (\mathbb{R}, 0) & \xleftarrow{h(x+y)} & (\mathbb{R}^2, 0) & \xrightarrow{\gamma} & (\mathbb{R}^2, 0) & \xrightarrow{f} & (\mathbb{R}, 0) \\ \parallel & & \hat{H} \downarrow & & \hat{K} \downarrow & & \parallel \\ (\mathbb{R}, 0) & \xleftarrow{X+Y} & (\mathbb{R}^2, 0) & \xrightarrow{\gamma} & (\mathbb{R}^2, 0) & \xrightarrow{f \circ \hat{K}^{-1}} & (\mathbb{R}, 0) \end{array}$$

where  $\hat{H}, \hat{K}$  are  $C^\infty$ -diffeomorphism germs. Therefore it is sufficient to show that  $C^0$ -commutativity implies  $C^\infty$ -commutativity in the following diagram:

(5.3)

$$\begin{array}{ccccccc} (\mathbb{R}, 0) & \xleftarrow{x+y} & (\mathbb{R}^2, 0) & \xrightarrow{\gamma} & (\mathbb{R}^2, 0) & \xrightarrow{f \circ \hat{K}^{-1}} & (\mathbb{R}, 0) \\ \parallel & & H \circ \hat{H}^{-1} \downarrow & & K \circ \hat{K}^{-1} \downarrow & & \downarrow k \\ (\mathbb{R}, 0) & \xleftarrow{X+Y} & (\mathbb{R}^2, 0) & \xrightarrow{\gamma} & (\mathbb{R}^2, 0) & \xrightarrow{\tilde{f}} & (\mathbb{R}, 0). \end{array}$$

We see that  $H \circ \hat{H}^{-1} = id$  on  $(\mathbb{R}^2, 0)$  and  $K \circ \hat{K}^{-1} = id$  on  $\{v \geq 0\}$ .

Now we consider the following two cases:

*Case  $\frac{\partial f}{\partial v}(0) < 0$ :* By the same argument as the case  $(III, I)^0$ , we see that  $k$  is a  $C^\infty$ -diffeomorphism germ. Define  $C^\infty$ -diffeomorphism germ  $\tilde{K} : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$  by  $\tilde{K}(u, v) = (u, \tilde{f}(u, v))^{-1} \circ (u, k \circ f \circ \hat{K}^{-1}(u, v))$ . Then we have  $\tilde{K} = id$  on  $\{v \geq 0\}$  and  $\tilde{f} \circ \tilde{K} = k \circ f \circ \hat{K}^{-1}$ . This completes the proof of rigidity in the case  $\frac{\partial f}{\partial v}(0) < 0$ .

*Case  $\frac{\partial f}{\partial v}(0) > 0$ :* In this case  $f^{-1}(0)$  is not in the 3-web domain  $\{v > 0\}$ . We see that  $f$  restricted to  $v$ -axis ( $v > 0$ ) is a  $C^\infty$ -diffeomorphism germ. Then by the similar argument as in  $(III, I)^0$  we see that  $k|_{\{t > 0\}}$  is a  $C^\infty$ -diffeomorphism germ. Moreover from (5.3) we have



$k(t) = \tilde{f}(0, \varphi(t))$  for  $t > 0$ , where  $\varphi(t)$  is the inverse function germ of  $t = f \circ \widehat{K}^{-1}(0, v)$ . Let  $\tilde{k}(t)$  be a natural  $C^\infty$ -extension of  $k|_{\{t>0\}}$  defined by  $\tilde{k}(t) = \tilde{f}(0, \varphi(t))$  for all  $t \in (\mathbb{R}, 0)$ . Clearly  $\tilde{k}$  is a  $C^\infty$ -diffeomorphism germ. Then similarly we can define  $C^\infty$ -diffeomorphism germ  $\tilde{K} : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$  by  $\tilde{K}(u, v) = (u, \tilde{f}(u, v))^{-1} \circ (u, \tilde{k} \circ f \circ \widehat{K}^{-1}(u, v))$ . By the same argument as in the case  $\frac{\partial f}{\partial v}(0) < 0$  we have the required result.

This completes the proof of rigidity of case (III, I)<sup>1</sup>. ■

**5.3. Proof of rigidity for (VI, I)**

In this case we suppose that  $\gamma = (x, y^3 + xy)$ . Denote by  $\Delta$  the set  $\{4u^3 + 27v^2 < 0\}$  of inner points of the cusp. At first we prepare a crucial Proposition.

**PROPOSITION 5.5.3.** *Let  $(g_i, \gamma, f_i)$  be two normal forms of type (VI, I), where  $g_i = y + \theta_i \circ \gamma, \theta_i \in \mathcal{M}_2 (i = 1, 2)$ . Then  $\theta_1|_\Delta = \theta_2|_\Delta$  and  $f_1|_\Delta = f_2|_\Delta$  if and only if there exist  $C^\infty$ -diffeomorphism germs  $H, K : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$  such that the following diagram commutes:*

$$\begin{CD} (\mathbb{R}, 0) @<g_1<< (\mathbb{R}^2, 0) @>\gamma>> (\mathbb{R}^2, 0) @>f_1>> (\mathbb{R}, 0) \\ @| @. @. @| \\ @. @. @. @. \\ @. @. @. @. \\ (\mathbb{R}, 0) @<g_2<< (\mathbb{R}^2, 0) @>\gamma>> (\mathbb{R}^2, 0) @>f_2>> (\mathbb{R}, 0) \end{CD}$$

**Proof.** Necessity: Assume that  $\theta_1|_\Delta = \theta_2|_\Delta$  and  $f_1|_\Delta = f_2|_\Delta$ . Define  $C^\infty$ -map germs  $H_i : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$  by  $H_i = (f_i \circ \gamma, g_i), i = 1, 2$ . Since the level curve  $f_i = 0$  and the tangent cone of the cusp are transversal, we see that  $H_i$  is a  $C^\infty$ -diffeomorphism germ. Namely  $H_i$  changes the 2-web  $(f_i \circ \gamma, g_i)$  to the canonical grid. Put  $H = H_2^{-1} \circ H_1$ . Then we have  $g_1 = g_2 \circ H$  and  $f_1 \circ \gamma = f_2 \circ \gamma \circ H$ .

On the other hand from the assumption clearly we have  $H|_{\gamma^{-1}(\Delta)} = id|_{\gamma^{-1}(\Delta)}$ . Then as was shown in [4]p.464 we see that  $H$  is a  $\gamma$ -lowable, namely there exists a  $C^\infty$ -diffeomorphism germ  $K : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$  such that  $K \circ \gamma = \gamma \circ H$ . Hence we see that these  $H, K$  realize the required commutativity.

Sufficiency is a corollary of the argument of the below. ■

Assume that two diagrams of (VI, I) are topologically equivalent:

$$\begin{CD} (\mathbb{R}, 0) @<g_1<< (\mathbb{R}^2, 0) @>\gamma>> (\mathbb{R}^2, 0) @>f_1>> (\mathbb{R}, 0) \\ @. @. @. @. \\ @. @. @. @. \\ @. @. @. @. \\ (\mathbb{R}, 0) @<g_2<< (\mathbb{R}^2, 0) @>\gamma>> (\mathbb{R}^2, 0) @>f_2>> (\mathbb{R}, 0) \end{CD}$$

where  $g_i = y_i + \theta_i \circ \gamma (i = 1, 2)$  and  $h, H, K, k$  are homeomorphism germs.

**Step 1.** *We may suppose that  $h = id$ .*

There is a 4-web structure in  $\Delta$  associated with (VI, I). Hence by rigidity of webs  $K|_\Delta$  is a  $C^\infty$ -diffeomorphism germ. Then as was shown in [4]p.469 (which is the similar argument

as in the previous cases) we see that  $h$  is a  $C^\infty$ -diffeomorphism germ. Hence  $(h \circ g_1, \gamma, f_1)$  is  $C^\infty$ -equivalent to  $(g_1, \gamma, f_1)$ . On the other hand by [4, Proposition 1 (p. 462)] we have the following  $C^\infty$ -commutative diagram

$$\begin{array}{ccccccc} (\mathbb{R}, 0) & \xleftarrow{h \circ g_1} & (\mathbb{R}^2, 0) & \xrightarrow{\gamma} & (\mathbb{R}^2, 0) & \xrightarrow{f_1} & (\mathbb{R}, 0) \\ \parallel & & \hat{H} \downarrow & & \hat{K} \downarrow & & \parallel \\ (\mathbb{R}, 0) & \xleftarrow{y + \theta \circ \gamma} & (\mathbb{R}^2, 0) & \xrightarrow{\gamma} & (\mathbb{R}^2, 0) & \xrightarrow{\hat{f} = f_1 \circ \hat{K}^{-1}} & (\mathbb{R}, 0) \end{array}$$

for some  $\theta \in \mathcal{M}_2$ , where  $\hat{H}, \hat{K}$  are  $C^\infty$ -diffeomorphism germs. Namely we have the following  $C^0$ -commutative diagram:

$$\begin{array}{ccccccc} (\mathbb{R}, 0) & \xleftarrow{y + \theta \circ \gamma} & (\mathbb{R}^2, 0) & \xrightarrow{\gamma} & (\mathbb{R}^2, 0) & \xrightarrow{\hat{f}} & (\mathbb{R}, 0) \\ \parallel & & H \circ \hat{H}^{-1} \downarrow & & K \circ \hat{K}^{-1} \downarrow & & \downarrow k \\ (\mathbb{R}, 0) & \xleftarrow{g_2} & (\mathbb{R}^2, 0) & \xrightarrow{\gamma} & (\mathbb{R}^2, 0) & \xrightarrow{f_2} & (\mathbb{R}, 0). \end{array}$$

Since the level curve  $\hat{f} = 0$  and the tangent cone of the cusp are also transversal, we can suppose that  $\hat{f}(u, 0) = u$  by the coordinate change, which is the inverse function of  $\hat{f}(u, 0)$ , in the target of  $\hat{f}$ . Hence it is sufficient to show the case  $h = id$ .

**Step 2.**  $\theta_1|_\Delta = \theta_2|_\Delta, f_1|_\Delta = f_2|_\Delta$  and apply Proposition 5.3.

By the same argument as in [4]p.470, due to  $h = id$  we see that  $H|_{\gamma^{-1}(\Delta)} = id|_{\gamma^{-1}(\Delta)}, K|_\Delta = id|_\Delta$ . Hence  $\theta_1 = \theta_2$  on  $\Delta$ . Moreover we have  $k(s) = s$  for  $s < 0$ . Hence  $f_1 = f_2$  on  $\Delta$ . Then applying the necessity part of Proposition 5.3, the proof of rigidity for  $(VI, I)$  is completed.  $\blacksquare$

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