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Comparison Results for Nonlinear Parabolic Equations with Monotone Principal Part

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In this work we prove an abstract comparison principle for abstract quasilinear parabolic problems with monotone principal part. This result is applied to parabolic problems having the $p-$Laplacian as principal part and with non-Lipschitz perturbations. April, 2000 ICMC-USP

1. INTRODUCTION

In this work we prove abstract comparison results for solutions of nonlinear parabolic problems with monotone principal part and apply these results to a parabolic problem having the $p-$Laplacian as principal part. To be more precise, let A be a maximal monotone operator in a Hilbert space H, $\{S(t); t \geq 0\}$ be the semigroup generated by $-A$ and, for $i = 0, 1$, consider the associated initial value problem

$$
\begin{aligned}
\dot{u} &= Au + f_i(t, u) \\
u(0) &= u_i.
\end{aligned}
\tag{1}
$$

Denote by $u(t, u_i, f_i)$ a solution of (1), $i = 0, 1$. Assume that H is endowed with an order (e.g, $u_0 \geq 0$ in $H = L^2(\Omega)$ if $u_0 \geq 0$ almost everywhere). A first comparison result in the ordered Hilbert space H states that $u_0 \geq u_1$ implies $S(t)u_0 \geq S(t)u_1$ for all $t \geq 0$. In this case we say that the semigroup $S(t)$ is increasing. We seek for conditions on the maximal monotone operator A that ensures this first comparison result. Once we have obtained

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conditions on A ensuring that $S(t)$ is increasing we extend this comparison result to the solutions of the problem (1).

The statement of the comparison results that we will present in this paper are similar to those already established elsewhere for semilinear parabolic equations. In particular the starting point of this theory is a result that can be stated in the following way: *If the resolvent of A is increasing (in a sense to be explained later) then, $S(t)$ is an increasing semigroup.* This result reads exactly the same as the corresponding result for the case when the maximal monotone operator A is linear. They differ, however, in the techniques used in the proof. This condition is actually a necessary and sufficient condition and its proof is based on a generalization of a result due to H. Brèzis that we state in the next section.

Concerning the comparison results for (1), the theorem that we would like to prove should read: *If $S(t)$ is increasing, $u_0 \geq u_1$ and $f_0(t,u) \geq f_1(t,u)$ for all t, u then $u(t, u_0, f_0) \geq u(t, u_1, f_1)$ as long as both solutions exist.* As we will see in the coming sections there are some technical difficultness that will impair us from obtaining such comparison result. These technical difficultness will lead to some additional assumptions on the vector fields f_i. The the comparison results that we will be able to prove for (1) will have statements similar to the statements of the comparison results presented in [1] for semilinear parabolic problems.

This paper is organized as follows. Section 2 is devoted to the proof of a preliminary result that will be the key to prove the abstract comparison results. Section 3 is devoted to the statement and proof of the abstract comparison results for the case of globally Lipschitz perturbations. Section 4 deals with the case of non-Lipschitz perturbations of subdifferentials. Section 5 is devoted to the application of these results to the study of a parabolic equation with the p-Laplacian perturbed by non-monotone non-Lipschitz operators. Finally, in Section 6 we exhibit some more examples of maximal monotone operators with increasing resolvent.

2. A PRELIMINARY RESULT

Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. Let A be a maximal monotone operator in H, and $S(t)$ be the semigroup generated by $-A$. We will employ the following notation.

- $J_\lambda := (I + \lambda A)^{-1}$, is the resolvent A;
- $A_\lambda := \dfrac{I - J_\lambda}{\lambda}$, is the Yosida approximation of A;
- $A^0 x$, is the least norm element of Ax.

Let φ be a convex, proper and lower semicontinuous (hereafter l.s.c) map defined in H. We define

$$\varphi_\mu(u) = \frac{1}{2\mu}|u - (I - \mu\partial\varphi)^{-1}u|^2 + \varphi((I - \mu\partial\varphi)^{-1}u).$$

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φ_μ thus defined is convex, Frechét-differentiable and $\partial\varphi_\mu = (\partial\varphi)_\mu$. Furthermore, $\varphi_\mu(x) \uparrow \varphi(x)$ as $\mu \downarrow 0$, $\forall\, x \in H$, (see Proposition 2.11 in [3], page 39).

The following theorem can be found in [3], Theorem 4.4, page 130.

THEOREM 2.1. *Let φ be a convex proper and l.s.c. map defined in H and assume that for any $x \in H$:*

(H-1) $\varphi(\mathrm{Proj}_{\overline{D(A)}}\, x) \leq \varphi(x)$.

Then, the following are equivalent:

$$
\begin{aligned}
&(i)\ \varphi((I+\lambda A)^{-1}x) \leq \varphi(x) && \forall\, x \in H, \forall\, \lambda > 0;\\
&(ii)\ \langle A_\lambda x, z\rangle\ \geq 0 && \forall\, z \in \partial\varphi(x), \forall\, \lambda > 0;\\
&(iii)\ \langle A_\lambda x, \partial\varphi_\mu(x)\rangle\ \geq 0 && \forall\, x \in H, \forall\, \lambda, \mu > 0;\\
&(iv)\ \langle A^0 x, \partial\varphi_\mu(x)\rangle\ \geq 0 && \forall\, x \in H, \forall\, \mu > 0;\\
&(v)\ \varphi_\mu(S(t)x) \leq \varphi_\mu(x) && \forall\, t \geq 0, \forall\, x \in \overline{D(A)}, \forall\, \mu > 0;\\
&(vi)\ \varphi(S(t)x) \leq \varphi(x) && \forall\, t \geq 0, \forall\, x \in \overline{D(A)}.
\end{aligned}
$$

Remark 2. 1. Note that, in the statement of Theorem 2.1, the operator A in general *is not* the subdifferential $\partial\varphi$ of φ.

The following theorem is a variation of Proposition 4.7, page 134, [3], and follows as a consequence of Theorem 2.1. The equivalent statements of the next theorem are the base to all comparison results that will be proved in this paper.

THEOREM 2.2. *Let φ be a convex proper and l.s.c. function in H and let A, B be maximal monotone operators in H satisfying*

(H-2) $\varphi(\mathrm{Proj}_{\overline{D(A)}}\, x - \mathrm{Proj}_{\overline{D(B)}}\, y) \leq \varphi(x-y), \forall\, x, y \in H$.

Then, the following are equivalent:

$$(i)\ \varphi((I+\lambda A)^{-1}x-(I+\lambda B)^{-1}y)\le\varphi(x-y) \qquad \forall\, x,y\in H,\forall\,\lambda>0;$$

$$(ii)\ \langle A_\lambda x-B_\lambda y,z\rangle\ \ge 0 \qquad \forall\, z\in\partial\varphi(x-y),\lambda>0;$$

$$(iii)\ \langle A_\lambda x-B_\lambda y,\partial\varphi_\mu(x-y)\rangle\ \ge 0 \qquad \forall\, x,y\in H,\forall\,\lambda,\mu>0;$$

$$(iv)\ \langle A^0 x-B^0 y,\partial\varphi_\mu(x-y)\rangle\ \ge 0 \qquad \forall\, x,y\in H,\forall\,\mu>0;$$

$$(v)\ \varphi_\mu(S_A(t)x-S_B(t)y)\le\varphi_\mu(x-y) \qquad \forall\, t,\mu>0,x,y\in\overline{D(A)};$$

$$(vi)\ \varphi(S_A(t)x-S_B(t)y)\le\varphi(x-y) \qquad \forall\, t>0,\forall\, x\in\overline{D(A)},y\in\overline{D(B)}.$$

Proof: Let $\mathcal{H}=H\times H$, be the Hilbert space with the inner product $([u,\bar u],[v,\bar v])=\langle u,\bar u\rangle+\langle v,\bar v\rangle$, and let $\mathcal{C}:\mathcal{H}\to\mathcal{H}$, be the operator defined by $\mathcal{C}[u,\bar u]=[Au,B\bar u]$. The operator $\mathcal{C}$ is monotone, since

$$\begin{aligned}(\mathcal{C}[u,\bar u]-\mathcal{C}[v,\bar v],[u,\bar u]-[v,\bar v]) &=\\ ([Au,B\bar u]-[Av,B\bar v],[u-v,\bar u-\bar v]) &=\\ ([Au-Av,B\bar u-B\bar v],[u-v,\bar u-\bar v]) &=\\ \langle Au-Av,u-v\rangle+\langle B\bar u-\bar v,\bar u-\bar v\rangle &\ge 0.\end{aligned}$$

Let us verify that $R(\mathcal{I}+\mathcal{C})=\mathcal{H}$. In fact, for all $[u,v]\in\mathcal{H}$ there exists $[x,y]\in\mathcal{H}$ such that

$$(I+A)x=u \qquad \text{e} \qquad (I+B)y=v,$$

then

$$\begin{aligned}(\mathcal{I}+\mathcal{C})[x,y] &= [x,y]+\mathcal{C}[x,y] = [x,y]+[Ax,By]\\ &= [x+Ax,y+By] = [(I+A)x,(I+B)y]=[u,v].\end{aligned}$$

Hence, $\mathcal{C}$ is maximal monotone in $\mathcal{H}$. Besides that we have:

1. $\mathcal{J}_\lambda[u,\bar u]=[J^A_\lambda u,J^B_\lambda\bar u]$, where $\mathcal{J}_\lambda$ is the resolvent of $\mathcal{C}$;
2. $\mathcal{C}_\lambda[u,\bar u]=[A_\lambda u,B_\lambda\bar u]$, where $\mathcal{C}_\lambda$ is the Yosida approximation of $\mathcal{C}$;
3. $\mathcal{S}(t)[u_0,\bar u_0]=[S_A(t)u_0,S_B(t)\bar u_0]$, where $\mathcal{S}(t)$ is the semigroup generated by $-\mathcal{C}$, $S_A(t)$ is the semigroup generated by $-A$ and $S_B(t)$ is the semigroup generated by $-B$.

Let φ be a convex, proper and l.s.c. function defined in H, and let ψ be the function given by $\psi[x,y]=\varphi(x-y)$. Then ψ is convex proper and l.c.s. and:

1. $\partial\psi[x,y] = \{[z,-z]; z \in \partial\varphi(x-y)\}$;
2. $\partial\psi_\lambda[x,y] = [(\partial\varphi)_{2\lambda}(x-y), -(\partial\varphi)_{2\lambda}(x-y)]$;
3. $\psi_\lambda[x,y] = (\varphi)_{2\lambda}(x-y)$.

With the considerations above the result now follows applying Theorem 2.1 to the operator $\mathcal{C}$ defined in $\mathcal{H}$, and the convex, proper and l.s.c. function ψ. □

Remark 2. 2. The hipothesis **(H-2)** is automatically satisfied if $\overline{D(A)} = H$ and $\overline{D(B)} = H$.

3. COMPARISON RESULTS

In this section we first prove abstract monotonicity results for semigroups of nonlinear contractions generated by maximal monotone operators using the results of previous section. Then, we use these monotonicity results to obtain comparison results for abstract nonlinear parabolic problems with monotone principal part and non-monotone perturbations of various types (time-dependent, globally Lipschitz and non-globally Lipschitz).

We start introducing a concept of order in a Banach space which is suitable for the desired comparison results:

Definition 3.1. An ordered Banach space is a pair $(X, \leq)$, where X is a Banach space and $\leq$ is a order relation in X satisfying:

1. $x \leq y$ implies $x+z \leq y+z,\ x,y,z \in X$;
2. $x \leq y$ implies $\lambda x \leq \lambda y,\ x,y \in X$ and $0 \leq \lambda \in \mathbb{R}$;
3. The positive cone $C = \{x \in X : x \geq 0\}$ is closed in X.

Definition 3.2. Let $(X, \leq)$ and $(Y, \preceq)$ be ordered Banach spaces. We say that a map $T : X \to Y$ is increasing if and only if $x \leq y$ implies $T(x) \preceq T(y)$. We say that T is positive if and only if $x \geq 0$ implies $T(x) \succeq 0$.

Let $(H, \leq)$ be an ordered Hilbert space and let A be a maximal monotone operator in H. Let C be the positive cone defined in H, and let I_C be the indicator map:

$$I_C = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{if } x \notin C. \end{cases}$$

I_C is a convex, proper and l.s.c. map, and therefore ∂I_C is a maximal monotone operator in H. In what follows we characterize $(I - \lambda\partial I_C)^{-1}$, $(\partial I_C)_\lambda$ and $(I_C)_\lambda$, for details see [3, 2].

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- the resolvent operator $(I - \lambda\partial I_C)^{-1}$ of ∂I_C is the projection in C. In other words, $y = (I - \lambda\partial I_C)^{-1}x \Leftrightarrow y = \mathrm{Proj}_C\, x$;
- the Yosida approximation of ∂I_C is given by $(\partial I_C)_\lambda = \dfrac{1}{\lambda}(x - \mathrm{Proj}_C\, x)$;
- $(I_C)_\lambda(x) = \dfrac{1}{2\lambda}|x - \mathrm{Proj}_C\, x|^2$.

A semigroup of nonlinear contractions $\{S(t) : t \geq 0\}$ is increasing if, for each $t \geq 0$, $S(t)$ is an increasing map. In what follows we give e necessary and sufficient condition so that the semigroup $\{S(t); t \geq 0\}$ generated by a maximal monotone operator $-A$ be increasing:

Theorem 3.1. *Let A be a maximal monotone operator in H satisfying*

(H-3) $I_C(\mathrm{Proj}_{\overline{D(A)}}\, x - \mathrm{Proj}_{\overline{D(A)}}\, y) \leq I_C(x - y), \forall\, x, y \in H.$

Then, the following are equivalent:

$$(i) \quad x \geq y \Rightarrow (I + \lambda A)^{-1}x \geq (I + \lambda A)^{-1}y \qquad \forall\, x, y \in H, \forall\, \lambda > 0;$$

$$(ii) \quad x \geq y \Rightarrow S(t)x \geq S(t)y \qquad \forall\, t \geq 0, \forall\, x, y \in \overline{D(A)}.$$

Furthermore, if $S(t)0 = 0$, $t \geq 0$ we have that

$$(iii) \quad x \geq 0 \Rightarrow S(t)x \geq 0 \qquad \forall\, t \geq 0, \forall\, x \in \overline{D(A)}.$$

and in this case we say that the semigroup is positive.

Proof: It is enough to consider $A = B$ and $\varphi = I_C$ in Theorem 2.2, and the result will follow as a consequence of the equivalence $(i) \Leftrightarrow (vi)$. ∎

More generally we have:

Theorem 3.2. *Under the hypothesis* **(H-3)**. *Denote by $u, \bar{u}$, the respective solutions of the following initial value problems:*

$$\begin{cases} \dfrac{d}{dt}u + Au \ni y \\ \\ u(0) = u_0, \end{cases} \qquad e \qquad \begin{cases} \dfrac{d}{dt}\bar{u} + A\bar{u} \ni \bar{y} \\ \\ \bar{u}(0) = \bar{u}_0. \end{cases}$$

If y, $\bar{y}$, u_0, $\bar{u}_0 \in H$, $y \geq \bar{y}$ and $u_0 \geq \bar{u}_0$, and if $(I + \lambda A)^{-1}$ is increasing, then $u(t) \geq \bar{u}(t)$ $\forall\, t > 0$.

Proof: In fact, it is enough to note that, if $y \in H$, the operator $A_y : H \to H$ given by $A_y u = Au - y$ is maximal monotone. Then the result follows from Theorem 2.2 with $\varphi = I_C$, $A = A_y$ e $B = A_{\bar{y}}$, observing that

$$w = (I + \lambda A_y)^{-1}u \Rightarrow w = (I + \lambda A)^{-1}(u + \lambda y),$$

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$$\bar{w} = (I + \lambda A_{\bar{y}})^{-1}\bar{u} \Rightarrow \bar{w} = (I + \lambda A)^{-1}(\bar{u} + \lambda\bar{y}),$$

in such a way that $u \geq \bar{u}$ e $y \geq \bar{y}$ implies $(I + \lambda A_y)^{-1}u \geq (I + \lambda A_{\bar{y}})^{-1}\bar{u}$, $0 < \lambda \in \mathbb{R}$, that is,

$$0 = I_C((I + \lambda A_y)^{-1}u - (I + \lambda A_{\bar{y}})^{-1}\bar{u}) \leq I_C(u - \bar{u}) = 0, \qquad u, \bar{u} \in H, \quad \lambda > 0. \quad \blacksquare$$

To extend the above results to the solutions of the non-homogeneous equations, we shall employ the same limiting procedure used to obtain existence of solutions for these equations.

THEOREM 3.3. *Let $T > 0$ and let $f(t), \bar{f}(t) \in L^1(0,T;H)$, $u_0 \in \overline{D(A)}$. Suppose that $u, \bar{u}$ are, respectively, the weak solutions of the following initial value problems:*

$$\begin{cases} \dfrac{d}{dt}u + Au \ni f(t) \\ \\ u(0) = u_0, \end{cases} \qquad \textit{and} \qquad \begin{cases} \dfrac{d}{dt}\bar{u} + A\bar{u} \ni \bar{f}(t) \\ \\ \bar{u}(0) = \bar{u}_0. \end{cases}$$

If A is a maximal monotone operator in H with increasing resolvent and satisfying the hypothesis **(H-3)**, *$f(t) \geq \bar{f}(t)$ a.e. in $[0,T]$, u_0 and $\bar{u}_0 \in H$, then $u_0 \geq \bar{u}_0$ implies $u(t) \geq \bar{u}(t)$ $\forall\, t \in [0,T]$.*

Proof: There are sequences $\{f_n\}$ and $\{\bar{f}_n\}$ of step functions in $[0,T]$ such that $f_n \to f$ and $\bar{f}_n \to \bar{f}$ in $L^1(0,T;H)$. We may assume, without loss of generality that for each $n \in \mathbb{N}$, given a partition of the interval $[0,T]$, $0 = t_0 < t_1 < ... < t_{kn} = T$, $f_n \equiv a_{n_i}$ and $\bar{f}_n \equiv \bar{a}_{n_i}$ in $[t_{i-1}, t_i]$ with a_{n_i}, $\bar{a}_{n_i} \in \mathbb{R}$, $a_{n_i} \geq \bar{a}_{n_i}$. Let u_n e $\bar{u}_n$ be strong solutions of the equations

$$\frac{d}{dt}u_n + Au_n \ni f_n \quad \text{and} \quad \frac{d}{dt}\bar{u}_n + A\bar{u}_n \ni \bar{f}_n$$

respectively, with $u_n(0) = u_0$, and $\bar{u}_n(0) = \bar{u}_0$. It follows from Theorem 3.2 that $u_n(t) \geq \bar{u}_n(t)$ $\forall\, t \in [0,T]$, $\forall\, n \in \mathbb{N}$. Since

$$|u_n(t) - u_m(t)| \leq \int_0^t |f_n(\tau) - f_m(\tau)| d\tau \qquad \forall\, t \in [0,T], \quad \text{and}$$

$$|\bar{u}_n(t) - \bar{u}_m(t)| \leq \int_0^t |\bar{f}_n(\tau) - \bar{f}_m(\tau)| d\tau \qquad \forall\, t \in [0,T],$$

then $u_n \to u$ and $\bar{u}_n \to \bar{u}$ uniformly in $C(0,T;H)$, and since the positive cone is closed, $u(t) \geq \bar{u}(t)$. □

Once we have the above result we start to compare solutions of non-linearly perturbed problems. As in the preceding proof the comparison is obtained through the same procedure used to obtain existence of solutions.

THEOREM 3.4. *Let A be a maximal monotone operator in an ordered Hilbert space H, and let $\leq$ be its order relation. Let $B, \bar{B}$ be globally Lipschitz maps in H, with Lipschitz constants ω and $\bar{\omega}$ respectively. Let u_0, $\bar{u}_0 \in \overline{D(A)}$, and let u, $\bar{u}$ be weak solutions of the problems*

$$\begin{cases} \frac{d}{dt}u(t) + Au(t) \ni Bu(t) \\ \\ u(0) = u_0, \end{cases} \qquad \text{and} \qquad \begin{cases} \frac{d}{dt}\bar{u}(t) + A\bar{u}(t) \ni \bar{B}\bar{u}(t) \\ \\ \bar{u}(0) = \bar{u}_0. \end{cases}$$

respectively. Suppose that A has increasing resolvent, satisfies the hypothesis **(H-3)**, *and there exists $G : \overline{D(A)} \to H$, increasing, such that $Bu \geq Gu \geq \bar{B}u$, $\forall\, u \in \overline{D(A)}$. Then $u_0 \geq \bar{u}_0$ implies $u(t) \geq \bar{u}(t) \quad \forall\, t \geq 0$.*

The proof of the above theorem is similar to the proof of the next theorem and therefore will be omitted.

THEOREM 3.5. *Let φ be a convex, proper and l.s.c. in the ordered Hilbert space H, let $\leq$ be its order. If $B, \bar{B}$ are maps from $[0,T] \times \overline{D(\varphi)}$ in H, each of them satisfying the conditions* (1) *and* (2) *of the* Proposition 3.13 *in* [3], page 107, *with Lipschitz constants ω e $\bar{\omega}$ respectively. Let u and $\bar{u}$ be solutions of the problems*

$$\begin{cases} \frac{d}{dt}u(t) + \partial\varphi(u(t)) \ni B(t,u(t)) \\ \\ u(0) = u_0, \end{cases} \qquad \text{and} \qquad \begin{cases} \frac{d}{dt}\bar{u}(t) + \partial\varphi\bar{u}(t) \ni \bar{B}(t,\bar{u}(t)) \\ \\ \bar{u}(0) = \bar{u}_0. \end{cases}$$

respectively. Suppose that $\partial\varphi$ has increasing resolvent, satisfies the hypothesis **(H-3)**, *and there exist $G : [0,T] \times \overline{D(\varphi)} \to H$, increasing in the variable u for each $t \in [0,T]$, and such that for all $t \in [0,T]$, $B(t,u) \geq G(t,u) \geq \bar{B}(t,u)$, $\forall\, u \in \overline{D(\varphi)}$. Then $u_0 \geq \bar{u}_0$ implies $u(t) \geq \bar{u}(t)$ $\forall\, t \geq 0$.*

Proof: We consider the sequences u_n e $\bar{u}_n$ defined by $u_0(t) \equiv u_0$ and $\bar{u}_0(t) \equiv \bar{u}_0$, u_n e $\bar{u}_n$ are solutions of the equations

$$\frac{d}{dt}u_n(t) + \partial\varphi(u_n(t)) \ni B(t, u_{n-1}(t))$$

$$u_n(0) = u_0,$$

and

$$\frac{d}{dt}\bar{u}_n(t) + \partial\varphi(\bar{u}_n(t)) \ni \bar{B}(t, \bar{u}_{n-1}(t))$$

$$\bar{u}_n(0) = \bar{u}_0.$$

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respectively. By Theorem 3.3 $u_1(t) \geq \bar{u}_1(t)$, $t > 0$. Assume by induction that $u_{n-1}(t) \geq \bar{u}_{n-1}(t)$, $t > 0$. Then by Theorem 3.3, $u_n(t) \geq \bar{u}_n(t)\ \forall\, n \in \mathbb{N}, t > 0$, and since

$$\begin{aligned} |u_{n+1}(t) - u_n(t)| &\leq \int_0^t |B(\tau, u_n(\tau)) - B(\tau, u_{n-1}(\tau))| d\tau \\ &\leq \omega \int_0^t |u_n(\tau) - u_{n-1}(\tau)| d\tau \end{aligned}$$

$$\begin{aligned} |\bar{u}_{n+1}(t) - \bar{u}_n(t)| &\leq \int_0^t |\bar{B}(\tau, \bar{u}_n(\tau)) - \bar{B}(\tau, \bar{u}_{n-1}(\tau))| d\tau \\ &\leq \bar{\omega} \int_0^t |\bar{u}_n(\tau) - \bar{u}_{n-1}(\tau)| d\tau \end{aligned}$$

then

$$|u_{n+1}(t) - u_n(t)| \leq \frac{(\omega T)^n}{n!} |u_1 - u_0|_{L^\infty(0,T;H)}$$

$$|\bar{u}_{n+1}(t) - \bar{u}_n(t)| \leq \frac{(\bar{\omega} T)^n}{n!} |\bar{u}_1 - \bar{u}_0|_{L^\infty(0,T;H)}$$

where $u_n \to u$ and $\bar{u}_n \to \bar{u}$ uniformly in $[0, T]$, in such a way that $u(t) \geq \bar{u}(t)$, $t > 0$. □

4. MORE GENERAL PERTURBATIONS OF SUBDIFFERENTIALS

This section is devoted to the proof of comparison results for problems with subdifferential principal part and for perturbations which are not globally Lipschitz. To accomplish that we will use the existence results proved in [9] which we briefly describe in Subsection 4.1. The proof of the comparison results are given in Subsection 4.2.

4.1. The Existence

In this subsection we state a particular case of a result due to Mitsuharu Ôtani, [9]. That result assures the existence of at least one strong solution for the equation

$$\frac{d}{dt} u(t) + Au(t) \ni Bu(t), \quad 0 < t < T, \tag{2}$$

where some of the restrictions are made on the operator A so that the hypotheses in the perturbation B may be relaxed. Since the proof of this result is used to prove our comparison results we shall indicate its main steps. We will assume that $A = \partial\varphi$, with φ satisfying:

(O-1) *φ is a convex, proper and l.s.c. from H into $[0, +\infty]$, such that for each $L \in (0, +\infty)$, the set $\{u \in H; \varphi(u) + |u|^2 \leq L\}$ is compact in H.*

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(**O-2**) *For all $u \in D(\partial\varphi)$, Bu is a convex subset of H;*

(**O-3**) *B is semi-closed in the following sense: for all interval $[a,b] \subset [0,T]$ the following holds: if $u_n \to u$ in $C(a,b;H)$, $\partial\varphi(u_n(\cdot)) \ni g_n(\cdot) \rightharpoonup g(\cdot) \in \partial\varphi(u(\cdot))$ in $L^2(a,b;H)$ and if $Bu_n(\cdot) \ni b_n(\cdot) \rightharpoonup b(\cdot)$ in $L^2(a,b;H)$, then $b(\cdot) \in Bu(\cdot)$; that is, $b(t) \in Bu(t)$ for almost all $[a,b]$;*

(**O-4**) *There exists a real, increasing and positive $\mathcal{L}_0(\cdot)$, a constant $\gamma_0 \in (0,1)$ and a constant c_0 such that:*

$$|||Bu|||_H^2 \leq \gamma_0 |\partial\varphi^0(u)|^2 + c_0\mathcal{L}_0(\varphi(u) + |u|), \quad \forall\, u \in D(\partial\varphi),$$

where $\partial\varphi^0(u)$ is the smallest norm element in $\partial\varphi(u)$ e $|||Bu|||_H = \ sup\{|b|, b \in Bu\}$.

In the proof of the local existence result the following version of Schauder-Tychonoff Theorem is used:

Theorem 4.1 (Browder). *Let K be a compact convex subset of a locally convex topological vector space X. Let T be a multi-valued l.s.c. map from K into X such that for each $x \in K$, $T(x)$ is a closed convex subset X such that $T(x) \cap K \neq \emptyset$. Then T has a fixed point in K; that is, there is an element $x_0 \in K$ such that $x_0 \in T(x_0)$.*

Definition 4.1. For each $u_0 \in \overline{D(\partial\varphi)}$, $h(t) \in L^2(0,T;H)$, and $S \in (0,T]$, we denote by $E_{u_0,S}(h)$ the unique strong solution of the equation

$$\frac{d}{dt}u(t) + \partial\varphi(u(t)) \ni h(t) \tag{3}$$

in $[0,S]$ satisfying $u(0) = u_0$. Let R be a fixed positive real number and let $K_{R,S}$ be the set

$$K_{R,S} := \{u \in L^2(0,S;H); |u|_{L^2(0,S;H)} \leq R\},$$

endowed with the weak topology of $L^2(0,S;H)$.

Let $\mathbb{B}_{u_0,S,R} : K_{R,S} \to K_{R,S}$ defined in the following manner: given $h \in K_{R,S}$, if there exist $b \in K_{R,S}$ such that $b(t) \in B(\mathbb{E}_{u_0,S}(h)(t))$ almost everywhere $(0,S)$, then

$$b \in \mathbb{B}_{u_0,S,R}(h),$$

and

$$D(\mathbb{B}_{u_0,S,R}) = \{h \in K_{R,S};\ \mathbb{B}_{u_0,S,R}(h) \neq \emptyset\}.$$

Lemma 4.1. *Suppose that the hypotheses* **(O-1), (O-2)** *and* **(O-3)** *are satisfied. Then, the graph $G(\mathbb{B}_{u_0,S,R})$, of $\mathbb{B}_{u_0,S,R}$, is closed in $K_{R,S} \times K_{R,S}$. Besides that, if $h \in D(\mathbb{B}_{u_0,S,R})$ then $\mathbb{B}_{u_0,S,R}(h)$ is a closed convex subset of $K_{R,S}$.*

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THEOREM 4.2. *Suppose that the hypotheses* **(O-1), (O-2), (O-3)** *and* **(O-4)** *are satisfied. If* $u_0 \in D(\varphi)$ *the problem* (2) *has a strong solution in* $[0, T_0]$ *for some* $T_0 \in (0, T]$.

Sketch of the Proof: The idea of the proof is to show that the operator $\mathbb{B}_{u_0,T_0,R}$ has at least one fixed point in K_{R,T_0} by making appropriate choice for the values of R and T_0. In that way, there exist $h \in K_{R,T_0}$ such that $\mathbb{B}_{u_0,T_0,R}(h) = h$ and, from the definition of $\mathbb{B}_{u_0,T_0,R}$ it follows that $\mathbb{E}_{u_0,T_0}(h)$ is a solution of the problem (2) in $[0, T_0)$.

4.2. Comparison

In what follows we are going to compare solutions of problems with perturbations more general then Lipschitz. As before, we are going to appeal to the techniques used to obtain the existence and Theorem 3.3. We assume throughout this section that $(\mathbf{H-3})$ is satisfied.

THEOREM 4.3. *Let* φ *satisfying the hypothesis* **(O-1)**, $u_0, \bar{u}_0 \in D(\varphi), u_0 \leq \bar{u}_0$. *Let* $B, \bar{B}$ *be two functions defined in* $D(\varphi)$ *taking values in* H *and satisfying the hypotheses* **(O-2)**, **(O-3)**, *and* **(O-4)**. *Suppose that* $\partial\varphi$ *has increasing resolvent, satisfies the hypothesis* **(H-3)**, *and that there exist* $G : D(\varphi) \to H$, *increasing, such that* $Bu \leq Gu \leq \bar{B}u$, $\forall\, u \in D(\varphi)$. *Then given a solution* $u(t)$ *of the problem*

$$\begin{cases} \dfrac{d}{dt}u(t) + \partial\varphi(u(t)) \ni Bu(t), \quad 0 < t < T, \\ \\ u(0) = u_0 \in D(\varphi), \end{cases} \tag{4}$$

there exists a solution $\bar{u}(t)$ *of the problem*

$$\begin{cases} \dfrac{d}{dt}\bar{u}(t) + \partial\varphi(\bar{u}(t)) \ni \bar{B}\bar{u}(t), \quad 0 < t < T, \\ \\ \bar{u}(0) = \bar{u}_0 \in D(\varphi), \end{cases} \tag{5}$$

$\bar{u}(t)$ *defined in* $[0, T_0)$ *for some* $T_0 > 0$, *and* $u(t) \leq \bar{u}(t)\, \forall\, t \in [0, T_0)$.

Proof: In fact, let $T_1 > 0$ such that $u(t)$ is solution of (4) in $[0, T_1)$, let $T_0 \leq T_1$ be a positive real number to be determined, and let K_{R,T_0}, $\mathbb{B}_{u_0,T_0,R}$ and $\bar{\mathbb{B}}_{\bar{u}_0,T_0,R}$ as in the Definition 4.1 with

$$R^2 = \frac{2\bar{\gamma}_0\varphi(\bar{u}_0) + \varepsilon_0}{(1 - \bar{\gamma}_0)},$$

where $\bar{\gamma}_0$ is as in the hypothesis **(O-4)**; that is

$$|||\bar{B}\bar{u}|||_H^2 \leq \bar{\gamma}_0|\partial\varphi^0(\bar{u})|^2 + \bar{c}_0\bar{\mathcal{L}}_0(\varphi(\bar{u}) + |\bar{u}|), \quad \forall\, \bar{u} \in D(\partial\varphi),$$

and ε_0 is an arbitrary positive number. By definition of $\mathbb{B}_{u_0,T_0,R}$ and u, $h := Bu$ is a fixed point $\mathbb{B}_{u_0,T_0,R}$ in K_{R,T_0}. We consider the restriction $\bar{\mathbb{B}}|_{\bar{K}}$ of $\bar{\mathbb{B}}_{\bar{u}_0,T_0,R}$ to the following subset of K_{R,T_0}:

$$\bar{K} = \{g \in K_{R,T_0}; g \geq h\}.$$

$\bar{K}$ is convex and compact. Let us verify that $\bar{\mathbb{B}}|_{\bar{K}} : \bar{K} \to \bar{K}$. Well, if $g \in \bar{K}$, then $g \geq h$ and by Theorem 3.3 $u_g(t) \geq u_h(t), t > 0$. Thus $\bar{B}(u_g) \geq G(u_g) \geq G(u_h) \geq B(u_h)$, and consequently $\bar{\mathbb{B}}(g) \geq \mathbb{B}(h) = h,\ \forall\, g \in \bar{K}$. It remains to conclude that given $g(t) \in \bar{K}$, if $\bar{b}(t) \in \bar{\mathbb{B}}(g(t))$ almost everywhere in $[0, T_0]$, then $|\bar{b}(t)|_{L^2(0,T_0;H)} \leq R$. For that we first observe that if $k \in \bar{K}$, and $v = \mathbb{E}_{\bar{u}_0,T_0}(k)$, then by Lemma 3.3 in [3], page 73, for almost all $t \in [0, T_0]$, we have:

$$\begin{aligned}\frac{d}{dt}\varphi(v(t)) &= \langle -\frac{d}{dt}v(t) + k(t), \frac{d}{dt}v(t)\rangle \\ &= -\,|-\frac{d}{dt}v(t) + k(t)|^2 + \langle -\frac{d}{dt}v(t) + k(t), k(t)\rangle,\end{aligned}$$

therefore

$$\begin{aligned}|-\frac{d}{dt}v(t) + k(t)|^2 + \frac{d}{dt}\varphi(v(t)) &\leq |-\frac{d}{dt}v(t) + k(t)|\,|k(t)| \\ &\leq \frac{1}{2}|-\frac{d}{dt}v(t) + k(t)|^2 + \frac{1}{2}|k(t)|^2,\end{aligned}$$

and therefore

$$\frac{1}{2}\int_0^t |-\frac{d}{dt}v(s) + k(s)|^2 ds + \varphi(v(t)) \leq \varphi(\bar{u}_0) + \frac{1}{2}\int_0^t |k(s)|^2 ds.$$

In one side

$$\int_0^t |-\frac{d}{dt}v(s) + k(s)|^2 ds \leq 2\varphi(\bar{u}_0) + \int_0^t |k(s)|^2 ds \leq 2\varphi(\bar{u}_0) + R^2, \tag{6}$$

and in another

$$\varphi(v(t)) \leq \varphi(\bar{u}_0) + \frac{1}{2}\int_0^t |k(s)|^2 ds \leq \varphi(\bar{u}_0) + \frac{1}{2}R^2.$$

That is

$$\max\{\varphi(v(t)); t \in [0, T_0]\} \leq C_0, \text{ where } C_0 = \varphi(\bar{u}_0) + \frac{1}{2}R^2. \tag{7}$$

Besides that, from the definition of subdifferential,

$$\begin{aligned}\varphi(v(t)) - \varphi(\bar{u}_0) &\leq \langle -\frac{d}{dt}v(t) + k(t), v(t) - \bar{u}_0\rangle \\ \Rightarrow\ \varphi(v(t)) - \varphi(\bar{u}_0) &\leq -\frac{1}{2}\frac{d}{dt}|v(t) - \bar{u}_0|^2 + \langle k(t), v(t) - \bar{u}_0\rangle\end{aligned}$$

$$\begin{aligned}\Rightarrow\ \frac{1}{2}\frac{d}{dt}|v(t)-\bar{u}_0|^2 &+\varphi(v(t)) \le \varphi(\bar{u}_0) + |k(t)||v(t)-\bar{u}_0| \\ \Rightarrow\ \frac{1}{2}|v(t)-\bar{u}_0|^2 &\le \int_0^t \varphi(\bar{u}_0)ds + \int_0^t |k(s)||v(s)-\bar{u}_0|ds.\end{aligned}$$

From which we conclude that

$$\begin{aligned}|v(t)-u(0)| &\le \sqrt{2t\varphi(\bar{u}_0)} + \sqrt{t}\left(\int_0^t |k(s)|^2 ds\right)^{1/2} \\ \Rightarrow |v(t)| &\le \sqrt{2t\varphi(\bar{u}_0)} + \sqrt{t}R + |\bar{u}_0|,\end{aligned}$$

and therefore

$$\max\{|v(t)|;\ 0 \le t \le T_0\} \le C_1,\ \text{where } C_1 = (\sqrt{2\varphi(\bar{u}_0)} + R)T_0^{1/2} + |\bar{u}_0|. \tag{8}$$

Now, if $\bar{b}(t) \in \bar{B}v(t)$, then by the hypothesis **(O-4)**, by (8) and by (7),

$$|\bar{b}(t)|^2 \le \bar{\gamma}_0 |\partial\varphi^0(v(t))|^2 + \mathcal{K},\ \text{where } \mathcal{K} = \bar{c}_0\bar{\mathcal{L}}_0(C_1 + C_0).$$

Hence

$$|\bar{b}(t)|^2_{L^2(0,T_0;H)} \le \bar{\gamma}_0 |\partial\varphi^0(v(t))|^2_{L^2(0,T_0;H)} + T_0\mathcal{K},$$

and by 6

$$|\bar{b}(t)|^2_{L^2(0,T_0;H)} \le \bar{\gamma}_0[2\varphi(\bar{u}_0) + R^2] + T_0\mathcal{K}.$$

From the choice of R,

$$|\bar{b}(t)|^2_{L^2(0,T_0;H)} \le (1-\gamma)R^2 - \varepsilon_0 + \gamma R^2 + T_0\mathcal{K}.$$

Choosing T_0 small enough so that

$$T_0\mathcal{K} \le \varepsilon_0,$$

we obtain

$$|\bar{b}(t)|_{L^2(0,T_0;H)} \le R.$$

That is, if $k \in \bar{K}$, $\bar{v} = \bar{\mathbb{E}}_{\bar{u}_0,T_0}(k)$, and $\bar{b}(t) \in \bar{B}v(t)$ almost everywhere in $[0,T_0]$, then $\bar{b} \in \bar{K}$. In this way

$$\bar{\mathbb{B}}_{\bar{u}_0,T_0,R}(\bar{K}) \subset \bar{K}.$$

Besides that $\bar{\mathbb{B}}|_{\bar{K}}$ satisfy all the hypotheses of the Theorem 4.1 since $\bar{\mathbb{B}}$ satisfy, from which one concludes by Lemma 4.1 and Theorem 4.1, that $\exists\ \bar{h} \in \bar{K}$ satisfying $\bar{h} = \bar{\mathbb{B}}_{\bar{u}_0,T_0,R}(\bar{h})$. Hence, there exists a solution $\bar{u}$ for the problem (5) in the interval $[0,T_0]$. Since $\bar{B}\bar{u}(t) \ge Bu(t)$ follows by Theorem 3.3 that $\bar{u}(t) \ge u(t)$ almost everywhere in $[0,T_0)$.□

Remark 4. 1. If in the above theorem we can ensure that the problem (5) has a *unique* solution $\bar{u}(t)$, then *all* solution $u(t)$ of the problem (4) satisfy $u(t) \leq \bar{u}(t), \forall\, t \in [0,T]$.

5. ONE APPLICATION

Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain with boundary $\partial\Omega$, and let $H = L^2(\Omega)$ be endowed with the following order relation: if $u, v \in L^2(\Omega)$ then

$$u \geq v \Leftrightarrow u(x) \geq v(x) \quad \text{a.e in } \Omega.$$

Consider the following second order nonlinear partial differential equation:

$$\frac{d}{dt}u - \Delta_p u = Bu, \quad p > 1, \tag{9}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian.

In this section we aim to apply the abstract results developed in the previous sections to obtain comparison results for (9). To that end we assume that the operator B is single valued and can be decomposed into the sum of two other operators, that is, $B = B_1 + B_2$, with B_1 and B_2 satisfying the following conditions:

(A-1) $-\Delta_p u - B_1(u)$ *is the subdifferential of a convex, proper and lower semicontinuous function* $\varphi : H \to \mathbb{R}$ *such that* $\varphi(u) \geq 0\, \forall\, u \in W_0^{1,p}(\Omega)$, $D(\varphi) = W_0^{1,p}(\Omega)$;

(A-2) B_1 *is the Nemitskiĭ operator associated to a function* $b_1 : \mathbb{R} \to \mathbb{R}$ *which is continuous and non-increasing and satisfies* $b_1(0) = 0$;

(A-3) B_2 *is the Nemitskiĭ operator associated to a function* $b_2 : \mathbb{R} \to \mathbb{R}$

Under these conditions we rewrite the equation (9) as:

$$\frac{d}{dt}u(t) + \partial\varphi(u(t)) = B_2(u(t)), \quad t > 0. \tag{10}$$

Assume that B_2 satisfies all the hypothesis required in Theorem 4.2. Then, given $u_0 \in W_0^{1,p}(\Omega)$ there exist $T > 0$ and at least one function $u \in C(0,T; L^2(\Omega))$ satisfying the equation (9) a.e. in $(0,T)$ and such that $u(0) = u_0$.

Note that $H = \overline{D(\varphi)} = \overline{D(\partial\varphi)}$; that is, $\partial\varphi$ is densely defined in H.

Let us show that the operator $Au = -\Delta_p u - B_1 u$ has increasing resolvent. In fact, we have

$$\begin{aligned}\langle Au - Av, (\partial I_C)_\mu (u-v)\rangle &= \int_\Omega (Au - Av)(u-v)^- dx \\ &= \int_{\Omega^-} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)(\nabla u - \nabla v) dx \\ &+ \int_{\Omega^-} (-B_1 u - (-B_1 v))(u-v) dx,\end{aligned}$$

where $\Omega^- = \{x \in \Omega; u(x) - v(x) < 0\}$. From hypothesis **(A-2)** and Lemma 4.4, [5], we can conclude that

$$\langle Au - Av, (\partial I_C)_\mu (u-v)\rangle \geq 0.$$

Then it follows from Theorem 2.2, $(iv) \Rightarrow (i)$, that $A = -\Delta_p - B_1$ has increasing resolvent.

Next consider the following auxiliary equation

$$\frac{d}{dt} w(t) + \partial\varphi(w(t)) = B_2^+(w(t)), \quad t > 0. \tag{11}$$

where for B_2^+ we assume that

(A-4) *B_2^+ is the Nemitskiĭ operator associated to a function $b_2^+ : \mathbb{R} \to \mathbb{R}$ which is continuous, non-decreasing, satisfies $b_2(s) \leq b_2^+(s)$ for all $s \in \mathbb{R}$ and is such that all conditions of Theorem 4.2 are satisfied for the problem 11*

As a consequence of the results in the previous sections we obtain the following result

THEOREM 5.1 (Comparison Theorem). *Assume* $(\mathbf{A-1})$, $(\mathbf{A-2})$,$(\mathbf{A-3})$ *and* $(\mathbf{A-4})$ *are satisfied and that $u_0, w_0 \in W_0^{1,p}(\Omega)$, $u_0 \leq w_0$. If a solution $u(t, u_0)$ of (10) is defined for $0 \leq t \leq T$ then there exists a solution $w(t, w_0)$ of (11) defined on $[0, T_0]$, $0 < T_0 \leq T$, such that $u(t, u_0) \leq w(t, w_0)$, $0 \leq t \leq T_0$.*

Remark 5. 1. This result is used to compare the solutions of (10) to equations with simpler right hand side B_2^+. In [4] this result is used to obtain global attractors for problems of the form (10) by comparing it to equations for which B_2^+ is given by $b_2^+(s) = c_1 s + c_2$, where c_1 and c_2 are constants. A similar result will also hold for comparison from bellow.

In what follows we make assumptions on the real valued functions b_1, b_2 and b_2^+ that ensure the conditions necessary to apply Theorem 5.1:

1. $p > n$, or
2. $p = n$ and b_1, b_2 and b_2^+ grow polynomially, or

3. $p < n$ and there are constants c_1, c_2 such that b_1, b_2 and b_2^+ satisfy the growth conditions
$|b_1(s)| + |b_2^+(s)| \leq c_1(|s|^{r_1-1} + 1),\ r_1 \leq \frac{np}{n-p}$;
$|b_2(s) - b_2(\bar{s})| \leq c_2|s - \bar{s}|(|s|^{r_2-1} + |\bar{s}|^{r_2-1} + 1),\ \forall\, s, \bar{s} \in \mathbb{R},\ r_2 \leq \min\{\frac{np}{2(n-p)}, \frac{n(p-1)+p}{n-p}\}$.

Remark 5. 2. Under the above assumptions on b_1 and b_2 it is not hard to see that all conditions of Theorem 4.2 are satisfied and therefore the initial value problem for (10) has at least one local solution. Concerning the initial value problem for (11), the above conditions on the growth and monotonicity of b_1 and b_2^+ ensure that it has a unique solution that depends continuously on the initial conditions. Besides uniqueness and continuity with respect to initial conditions we also have that the solutions of (11) converge to an equilibrium for (11), that is, a solution of $\partial\varphi(\eta) - B_1(\eta) - B_2^+(\eta) = 0$. All these conclusions follow from the fact that $\partial\varphi - B_1 - B_2^+$ is the subdifferential of a convex, proper and lower semicontinuous function and from basic results on the theory of differential equations with maximal monotone principal part (see, [2, 3, 10]).

6. MORE EXAMPLES OF INCREASING RESOLVENT OPERATORS

In this section we give some other examples of operators which have increasing resolvent. Equations having one of these operators as main part may have comparison results similar to the ones developed in the previous sections. One can verify that these operators have increasing resolvent in the same way we have shown the p-Laplacian operator has increasing resolvent. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain with boundary $\partial\Omega$.

EXAMPLE 6.1. Consider $H = H^{-1}(\Omega)$ endowed with the inner product given by the extension to $H \times H$ of the bilinear form

$$\langle u, v \rangle = \left(u, (-\Delta_D)^{-1} v\right)_{L^2(\Omega)}, u, v \in L^2(\Omega),$$

where Δ_D denotes the Laplacian operator with homogeneous Dirichlet boundary condition. For $p \geq 2$, and $V = L^p(\Omega)$, we have

$$V \subset H \subset V'$$

with continuous and dense inclusions. Let $A_0 : V \to V'$ be the operator defined by the form $a(\cdot,\cdot)$, given by

$$a(u, v) = \frac{1}{p-1} \int_\Omega |u|^{p-2} u v dx.$$

(see [8], Ex. 3.2, p.191).

EXAMPLE 6.2. Consider the operator A_1 given by

$$A_1 u = \sum_1^n \frac{\partial}{\partial x_i}\left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right).$$

The operator A_1 is the subdiferential $\partial\phi$ of a convex, proper, l.s.c application ϕ defined on $H = L^2(\Omega)$,

$$\phi(u) = \begin{cases} \dfrac{1}{p} \displaystyle\sum_1^n \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^p dx, & \text{if } u \in W_0^{1,p}(\Omega) \\ +\infty, & \text{otherwise.} \end{cases}$$

EXAMPLE 6.3. Let A_2 be the operator given by

$$A_2 u = -\text{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u),$$

where $a \in C^1(\mathbb{R}_+)$ such that $a'(\sigma) \geq 0$, $\sigma \in \mathbb{R}_+$ and $\delta\sigma^{p-1} \geq a(\sigma^p)\sigma^{p-1} \geq \alpha\sigma^{p-1} + \beta$, with $\delta, \alpha, \beta \geq 0$.

The operator A_2 is the subdiferential $\partial\phi$ of a convex, proper, l.s.c application ϕ defined on $H = L^2(\Omega)$,

$$\phi(u) = \begin{cases} \dfrac{1}{p} \displaystyle\sum_1^n \int_\Omega A(|\nabla u|^p) dx, & \text{if } u \in W_0^{1,p}(\Omega) \\ +\infty, & \text{otherwise,} \end{cases}$$

where $A(u) = \displaystyle\int_0^u a(\sigma) d\sigma$.

The operator A_0 satisfies the conditions of Theorem 3.4, and the operators A_1 and A_2 satisfy the conditions of Theorem 4.3 with perturbations satisfying the same growth conditions considered in the previous section.

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