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Asymptotic Behaviour of Nonlinear Parabolic Equations with Monotone Principal Part

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We consider nonlinear parabolic equations with subdifferential principal part and give conditions under which they posses global attractors in spite of considering non-Lipschitz perturbations. The case of globally Lipschitz perturbations of a maximal monotone operator has been addressed in [4]. In the case of perturbations which are not globally Lipschitz, the main difficultness is the lack of uniqueness of solutions which at first does not even allow us to define attractors. We overcome this difucultness for problems enjoying certain regularity and absorption properties that allow uniqueness of solutions after some time has been elapsed. The results developed here are applied to the case when the subdifferential operator is the $p-$Laplacian to obtain existence of attractors and the existence of periodic solutions. April, 2000 ICMC-USP

1. INTRODUCTION

In this paper we consider the existence of an asymptotic set of states (global compact attractor, as in [7]) for a problem of the form

$$\begin{aligned} &\frac{d}{dt}u + Au = Bu, \\ &u(0) = u_0, \end{aligned} \tag{1}$$

where A is a maximal monotone operator and B is a non-monotone, non-globally Lipschitz operator defined in a dense subset of a Hilbert space H which contains $D(A)$. In [4] the authors prove the existence of a global attractors for the semigroup associated to

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(1) in $\overline{D(A)}$ when $B : H \to H$ is globally Lipschitz. In the case of Lipschitz perturbations of maximal monotone operators there is a well established theory that ensures global existence, uniqueness and continuity with respect to initial data and therefore associates a continuous semigroup to the differential equation, see [2, 3, 12] and references there in. This allow us to employ the existing abstract theory to obtain existence of global attractors (see [4]). In the case of non-Lipschitz perturbations the existence of solutions is proved in [11] by using a Schauder-Tychonoff-type fixed point theorem, but only for the case when $A = \partial\varphi$ is the subdifferential of a convex proper lower semicontinuous (hereafter we will write l.s.c. for lower semicontinuous) map $\varphi : H \to \mathbb{R}$ and for initial data u_0 in the domain of φ; that is $u_0 \in D(\varphi)$ (in fact, in [11] the author considers initial conditions also in larger spaces but we will only use the existence results for initial data in $D(\varphi)$). This fixed point theorem will not allow uniqueness of solutions and in fact, in general, we do not know whether uniqueness can be proved or not. In spite of this we wish to be able to define the asymptotic dynamics of (1) for the case of non-Lipschitz perturbations B. It is clear that we will need to adapt the existing abstract theory to encompass this case.

Since we are unable to ensure uniqueness of solutions to (1) for non-Lipschitz, non-monotone perturbations B, we are unable to associate a semigroup to it. To start overcoming this difficultness we will need to consider a family multi-valued operators $\{V_t\}$ in a Banach space $\mathcal{X}$, $V_t : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ that enjoy the semigroup property. This family will be called pseudo-semigroup. For this family we define the concept attraction, invariance and, under some additional hypothesis, we prove the existence of a global attractor.

To obtain the existence of attractors we assume the following strong uniform dissipation and smoothing hypothesis: Let $\{V_t\}$ be a pseudo-semigroup and $\{T_t\}$ be a continuous semigroup. We will say they satisfy Hypothesis **H-R** if:

H-R: For each bounded subset $\mathcal{M} \subset \mathcal{X}$ there exists $\tau_0 = \tau_0(\mathcal{M}) > 0$, such that if $\tau > \tau_0$ and $x_\tau \in V_\tau(\mathcal{M})$ then for all $t > 0$, $V_t(x_\tau) = T_t(x_\tau)$.

Note that this hypothesis in particular is saying that we have uniqueness of solutions for initial data belonging to a point far (in time) into the orbit. In the applications this will be associated to strong regularity and attraction properties for the principal part of the parabolic problem.

Once we have established the abstract results we apply them to the case when the monotone operator is the $p-$Laplacian. The main tool that we employ, to ensure the needed smoothing and attraction properties in this application, is a comparison result developed in [5].

There is a vast literature on existence of asymptotic set of states (attractors) for evolution differential problems with linear principal part (enjoying uniqueness and continuity with respect to initial conditions), see [8, 7, 13] and references there in. The problems with nonlinear principal part have been left aside except for a few isolated efforts. One of the reasons for this is that the lack of uniqueness and continuity with respect to initial conditions seems to be a unavoidable barrier. In the class of problems with nonlinear principal part, the problems that seem to have enough structure to have a well organized theory on existence of attractors are those in the class of problems that enjoy enough regularity to ensure uniqueness after some time has been elapsed. In this class we distinguish some

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semigroups generated by differential problems with subdifferential principal part, since in this case the system has a natural energy that can be used to obtain a priori bounds and some regularity results are available.

This paper is organized into four sections which we briefly describe next. In Section 2 we state and prove results concerning the existence of attractors for problems lacking uniqueness but enjoying a strong smoothing and attraction condition condition. In Section 3 consider a parabolic equation for which the principal part is the $p-$Laplacian and the perturbation is not globally Lipschitz. To obtain the existence of a global attractor we first, in Subsection 3.1, use comparison results to obtain a strong smoothing and attraction property and then, in Subsection 3.2, we apply the strong smoothing and attraction properties obtained in Subsection 3.1 and the results of Section 2, in Subsection 3.3 we consider examples of perturbations that satisfies the conditions of Subsections 3.1 and 3.2, in Subsection 3.4 we introduce a periodic perturbation to the equation and obtain the existence of periodic orbits. In Section 4 (Appendix) we prove two auxiliary results, first we prove that when the principal part of the parabolic equation is the $p-$Laplacian (Dirichlet boundary condition) and the perturbation is globally Lipschitz we have that there exists a global attractor in $W_0^{1,p}(\Omega)$ and second we prove that for certain values of p the domain of the $p-$Laplacian is embedded in $L^\infty(\Omega)$.

2. ATTRACTORS FOR PROBLEMS LACKING UNIQUENESS

Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space, and let $\{V_t\} = \{V_t; t \in \mathbb{R}^+\}$ be a family of nonlinear operators satisfying

- $V_t : \mathcal{X} \to \mathcal{P}(\mathcal{X})$, $\forall\, t \geq 0$;
- If $\mathcal{M} \subset \mathcal{X}$, $V_t(\mathcal{M}) = \bigcup_{x\in\mathcal{M}} V_t(x)$, $\forall t \geq 0$;
- $V_0(x) = x$; $\forall\, x \in \mathcal{X}$;
- $V_{t_1+t_2}(x) = V_{t_1}(V_{t_2}(x))$, $\forall\, t_1, t_2 \in \mathbb{R}^+$, $\forall\, x \in \mathcal{X}$.

Under these conditions we say $\{V_t\}$ is a pseudo-semigroup. As it is done for semigroups we can define the following:

Definition 2.1. *Let $\mathcal{A}$ and $\mathcal{M}$ be subsets of $\mathcal{X}$. We say that $\mathcal{A}$* attracts *$\mathcal{M}$ under $\{V_t\}$ or $\mathcal{M}$* is attracted to *$\mathcal{A}$ under $\{V_t\}$, if for all $\varepsilon > 0$ there exists $T_1(\varepsilon, \mathcal{M}) \in \mathbb{R}^+$ such that $V_t(\mathcal{M}) \subset \mathcal{O}_\varepsilon(\mathcal{A})$ for all $t > T_1(\varepsilon, \mathcal{M})$.*

Definition 2.2. *A subset $\mathcal{M} \subset \mathcal{X}$ is called* invariant *under $\{V_t\}$ if $V_t(\mathcal{M}) = \mathcal{M}$. We say that an invariant subset $\mathcal{M}$ is* maximal compact invariant *if all compact invariant subset relative to $\{V_t\}$ is contained in $\mathcal{M}$.*

Definition 2.3. *An invariant set $\mathcal{A}$ relative to $\{V_t\}$ is a* global attractor *for $\{V_t\}$ if $\mathcal{A}$ is a maximal compact invariant set that attracts each bounded subset $\mathcal{M} \subset \mathcal{X}$ under $\{V_t\}$.*

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We can define also the ω-limit set:

Definition 2.4. *Let us suppose $\mathcal{M} \subset \mathcal{X}$. We say that y belongs to* ω-limit set of $\mathcal{M}$, *$\omega(\mathcal{M})$, if $y = \lim_{k\to\infty} y_k$, $y_k \in V_{t_k}(\mathcal{M})$, $t_k \to +\infty$. We denote $\omega(x)$ the ω-limit set of $\{x\}$, $\forall\, x \in \mathcal{X}$.*

Theorem 2.1. *Let $\{V_t\}$ be a pseudo-semigroup and $\{T_t\}$ a continuous and compact semigroup in $\mathcal{X}$ satisfying* **H-R**. *If $\mathcal{M}_0 \subset \mathcal{X}$ is such that*

$$\{y \in \mathcal{X}; y \in V_t(\mathcal{M}_0), t > \tau(\mathcal{M}_0)\} = \bigcup_{t>\tau(\mathcal{M}_0)} V_t(\mathcal{M}_0)$$

is a bounded subset of $\mathcal{X}$, where $\tau(\mathcal{M}_0)$ is given in **H-R**, *then we have*

- *$\omega(\mathcal{M}_0)$ is non-empty, compact and invariant under $\{T_t\}$;*
- *$\omega(\mathcal{M}_0)$ attracts $\mathcal{M}_0$;*
- *$\omega(\mathcal{M}_0)$ is the minimal closed set that attracts $\mathcal{M}_0$.*

Proof: Denote by ω_V and ω_T the ω-limit sets relatives to V_t and T_t respectively. It is enough to note that if $\mathcal{M}_0$ is a bounded subset of $\mathcal{X}$ then $\omega_V(\mathcal{M}_0) = \omega_V(V_{\tau(\mathcal{M}_0)}(\mathcal{M}_0)) = \omega_T(V_{\tau(\mathcal{M}_0)}(\mathcal{M}_0))$. Then follows from properties of ω-limit sets for continuous semigroups (see [8]) that $\omega_V(\mathcal{M}_0) = \omega_T(V_{\tau(\mathcal{M}_0)}(\mathcal{M}_0))$ is non-empty and compact (invariant under $\{T_t\}$) and is also the minimal closed set that attracts $\mathcal{M}_0$ under $\{T_t\}$ and therefore under $\{V_t\}$. ∎

Note that $\omega(\mathcal{M}_0)$ is invariant under $\{T_t\}$ but we may not assure the invariance under $\{V_t\}$. The invariance under $\{V_t\}$ will be assumed in this abstract setting to get the existence of attractors. In the applications this invariance will be granted from the construction of $\{T_t\}$ and from uniform bounds on the $\omega-$limit sets.

Now we can prove the following:

Theorem 2.2. *Let $\{V_t\}$ be a pseudo-semigroup and $\{T_t\}$ a continuous compact and bounded dissipative semigroup in $\mathcal{X}$ satisfying* **H-R**. *Assume also that, if $\mathcal{M} \subset \mathcal{X}$ is a bounded subset of $\mathcal{X}$, then the ω-limit set $\omega_V(\mathcal{M})$ is invariant under $\{V_t\}$. Then there exists an attractor $\mathcal{A}$ for $\{V_t\}$ in $\mathcal{X}$.*

Proof: In fact, since $\{T_t\}$ and $\{V_t\}$ satisfy **H-R** and $\{T_t\}$ is bounded dissipative, there exists a bounded $\mathcal{M}_0 \subset \mathcal{X}$ which attracts each bounded subset of $\mathcal{X}$ under $\{V_t\}$. Let be $\varepsilon_1 > 0$. Then $\mathcal{O}_{\varepsilon_1}(\mathcal{M}_0)$ is an absorving bounded subset, that means, for all bounded subset $\mathcal{M} \subset \mathcal{X}$, there exists $\mathcal{T}(\mathcal{M}) > 0$ such that if $t > \mathcal{T}(\mathcal{M})$, $V_t(\mathcal{M}) \subset \mathcal{O}_{\varepsilon_1}(\mathcal{M}_0)$. Let be $\mathcal{M}_1 := \mathcal{O}_{\varepsilon_1}(\mathcal{M}_0)$. We have that $V_t(\mathcal{M}_1) \subset \mathcal{O}_{\varepsilon_1}(\mathcal{M}_0)$ if $t > \mathcal{T}(\mathcal{M}_1)$. Since $\mathcal{M}_1$ is bounded, we can conclude that $\omega(\mathcal{M}_1)$ is a compact non-empty subset of $\mathcal{X}$, and also $\omega(\mathcal{M}_1)$ is invariant and attracts $\mathcal{M}_1$ under $\{V_t\}$. That means for $\varepsilon > 0$ there exists $t_1(\varepsilon)$ such that $V_t(\mathcal{M}_1) \subset \mathcal{O}_\varepsilon(\omega(\mathcal{M}_1))$ for all $t > t_1(\varepsilon)$. Then if $\mathcal{M} \subset \mathcal{X}$ is bounded, $V_t(\mathcal{M}) \subset \mathcal{O}_\varepsilon(\omega(\mathcal{M}_1))$

for $t > t_1(\varepsilon) + \mathcal{T}(\mathcal{M})$. So $\omega(\mathcal{M}_1)$ is an invariant compact set which attracts each bounded subset of $\mathcal{X}$ under $\{V_t\}$. Also if $\mathcal{K}$ is an invariant subset of $\mathcal{X}$ under $\{V_t\}$ then $\mathcal{K} \subset \omega(\mathcal{M}_1)$ since $\omega(\mathcal{K}) = \mathcal{K}$ is the minimal closed set which attracts $\mathcal{K}$. Thus $\mathcal{K} = \omega(\mathcal{K}) \subset \omega(\mathcal{M}_1)$ and $\mathcal{A} = \omega(\mathcal{M}_1)$ is a global attractor. ∎

We note that the above result can be extended to the case when the semigroup $\{T_t\}$ is point dissipative and compact or to the case when orbits of bounded sets are bounded and the semigroup is point dissipative and asymptotically compact. These extensions are direct consequence of the existing results for continuous semigroups and of the arguments above and will not be reproduced here.

For the remaining of this section we will have in mind a non-autonomous nonlinear parabolic differential equations for which the principal part is autonomous and maximal monotone and the perturbation is nonlinear and time dependent. Our concerns will be directed to the case when the perturbation is time periodic.

We start defining *pseudo-evolution process*. A family $\{V_{t,t_0} : \mathcal{X} \to \mathcal{P}(\mathcal{X}) : t \geq t_0 \geq 0\}$ is called a pseudo-evolution process if the following properties hold

- $V_{t_0,t_0}(x) = x$, $\forall\, x \in \mathcal{X}$, $t_0 \geq 0$;
- $V_{t,t_0}(x) = V_{t,\tau}(V_{\tau,t_0}(x))$, $\forall\, t \geq \tau \geq t_0 \geq 0$ and $\forall\, x \in \mathcal{X}$,

where, for $\mathcal{M} \subset \mathcal{X}$, $V_{t,t_0}(\mathcal{M}) = \cup_{x\in\mathcal{M}} V_{t,t_0}(x)$, $t \geq t_0 \geq 0$. We say that a pseudo-evolution process is $r-$periodic if $V_{r+\sigma,r+\tau} = V_{\sigma,\tau}$ for any $\sigma \geq \tau \geq 0$. If V_{t,t_0} is a single valued operator for each $t \geq t_0 \geq 0$ we say that V_{t,t_0} is an *evolution process* and in this case, if $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathcal{X} \ni (t, t_0, x) \mapsto V_{t,t_0}(x) \in \mathcal{X}$ is a continuous map we say that V_{t,t_0} is a continuous evolution process. If for any $t_0 \geq 0$ and bounded subset $\mathcal{M}$ of $\mathcal{X}$ the set $V_{t,t_0}(\mathcal{M})$ is relatively compact we say that the evolution process V_{t,t_0} is compact.

We say that a pseudo-evolution process V_{t,t_0} is bounded dissipative if there is a bounded subset $\mathcal{M}_0$ of $\mathcal{X}$ with the property that, given $t_0 \geq 0$ and bounded subset $\mathcal{M}$ of $\mathcal{X}$, there exists $t_1 = t_1(t_0, \mathcal{M})$ such that $V_{t,t_0} \subset \mathcal{M}_0$, $\forall t \geq t_1$.

We now state a condition for pseudo-evolution processes which is analogous to the condition **H-R**. A pseudo-evolution process V_{t,t_0} is said to satisfy the condition **NH** if there is a continuous compact evolution process T_{t,t_0} such that

NH: For each $t_0 \geq 0$ and bounded subset $\mathcal{M} \subset \mathcal{X}$ there exist $\tau_0 = \tau_0(t_0, \mathcal{M}) > 0$ such that if $\tau > \tau_0$ and $x_\tau \in V_{\tau,t_0}(\mathcal{M})$ then, for all $t > \tau$, $V_{t,\tau}(x_\tau) = T_{t,\tau}(x_\tau)$.

Proceeding as in the continuous case we can define attraction and $\omega-$limit for the discrete semigroup $\mathcal{V}_n = \{V_{nr+\sigma,\sigma}; n \geq 0\}$. Denote by $\mathcal{T}_n = \{T_{nr+\sigma,\sigma}; n \geq 0\}$ a discrete compact semigroup associated to the continuous compact $r-$periodic evolution process T_{t,t_0}. As a consequence of **NH** we have that for any bounded subset $\mathcal{M}$ of $\mathcal{X}$ there is a large enough integer $n_0 \geq 0$ such that $\mathcal{V}_n x_0 = \mathcal{T}_n x_0$, for all $x_0 \in V_{n_0}(\mathcal{M})$, $n \geq 0$. With this in mind we can prove the following result

THEOREM 2.3. *Let V_{t,t_0} be an $r-$periodic pseudo-evolution process satisfying the condition* **NH**. *Assume that, if $\mathcal{M}$ is a bounded subset of $\mathcal{X}$, then the $\omega-$limit set of $\mathcal{M}$ under the discrete semigroup $\mathcal{V}_n$ is invariant unde $\mathcal{V}_n$. Then, there is an attractor for the discrete semigroup $\mathcal{V}_n$ and as a consequence of that there is a fixed point for $V_{r+\sigma,\sigma}$.*

The proof of this theorem follows from the considerations in this section and from the results in [7], Section 3.6.

3. APPLICATIONS

Let H be a Hilbert space and $\| \cdot \|_H$ its norm. In this section we consider problems of the form (1) for the case when A is the subdifferential of a convex, proper and l.s.c. map $\varphi : H \to \mathbb{R}$. Next we state a set of hypothesis that will be needed to ensure local and global existence, to allow strong smoothing and absorption properties and to ensure the existence of global attractors. The first five hypotheses are taken from [11] and will be used to obtain global existence and the remaining hypotheses are related to the existence of attractors for the case when the subdifferential operator is the $p-$Laplacian. We assume throughout this section that φ and B satisfy:

H-1 *φ is a convex proper l.s.c. application from H to $[0, +\infty]$ and for each $L \in (0, +\infty)$ the set $\{u \in H; \varphi(u) + \|u\|_H^2 \leq L\}$ is compact in H;*

H-2 *For all $u \in D(\partial\varphi)$, Bu is a convex subset of H;*

H-3 *B is demi-closed in the following sense: for each interval $[a, b] \subset [0, T]$ we have that, if $u_n \to u$ in $C(a, b; H)$, $g_n \rightharpoonup g$ in $L^2(a, b; H)$ with $g_n(t) \in \partial\varphi(u_n(t))$, $g(t) \in \partial\varphi(u(t))$ a.e. in $[a, b]$, and if $b_n \rightharpoonup b$ in $L^2(a, b; H)$ with $b_n(t) \in Bu_n(t)$ a.e. in $[a, b]$, then $b(t) \in Bu(t)$ a.e. in $[a, b]$;*

H-4 *There is a real increasing positive function $\mathcal{L}_0(\cdot)$, $\gamma \in (0, 1)$ and $c \in \mathbb{R}$ such that:*

$$|||Bu|||_H^2 \leq \gamma \|\partial\varphi^0(u)\|_H^2 + \mathcal{L}_0(\|u\|_H)[\{\varphi(u)\}^2 + c], \quad \forall\, u \in D(\partial\varphi),$$

where $\partial\varphi^0(u)$ is the minimal section in $\partial\varphi(u)$ and $|||Bu|||_H = \ sup\{\|b\|_H, b \in Bu\}$.

H-5 *There are positive constants α and β such that*

$$\langle v + b, u\rangle + \alpha\varphi(u) \leq \beta(\|u\|_H^2 + 1) \qquad \forall\, v \in -\partial\varphi(u), b \in Bu \text{ and } u \in D(\partial\varphi).$$

Next we restrict our attention to the case when $\partial\varphi$ is the $p-$Laplacian. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and let $H = L^2(\Omega)$. Consider the following nonlinear second order partial differential equation

$$\frac{d}{dt}u - \Delta_p u = Bu, \quad p > 2, \tag{2}$$

subjected to homogeneous Dirichlet boundary condition. Let us suppose that B is a Nemitskiĭ operator associated to some real function $\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2$ and denote B_1 and B_2 the Nemitskiĭ operators associated to $\mathbf{b}_1$ and $\mathbf{b}_2$, respectively. Actually, we suppose

H-6 $-\mathbf{b}_1$ *is increasing and* $\mathbf{b}_2(r) \geq kr + k_0$ *for all* $r \in \mathbb{R}$;

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H-7 $\mathbf{b}_1(0) = 0$*;*

H-8 *there exists* c_1 *such that for all fixed* $\delta \in \mathbb{R}$ *and some* $c_2(\delta)$*,* $\mathbf{b}_2(r) \leq c_1 r + c_2(\delta)$*,* $r \geq -\delta$*;*

H-9 $\mathbf{b}_2$ *satisfies a local Lipschitz condition, what means that if* $\mathcal{K} := \{u \in H; \|u\|_{L^\infty(\Omega)} \leq \eta\}$ *for some* $\eta > 0$*, then there exists* $\omega = \omega(\eta) > 0$*, such that* $\|B_2(u) - B_2(v)\|_{L^2(\Omega)} \leq \omega\|u - v\|_{L^2(\Omega)}\ \forall\, u, v \in \mathcal{K}$*.*

Remark 3. 1. Note that if k is negative we may incorporate kr to b_1 and assume that k is zero. Therefore we may always assume that k and c_1 are non-negative.

Under these conditions we rewrite equation (2):

$$\frac{d}{dt}u(t) + \partial\phi(u(t)) = B_2(u(t)), \quad t > 0. \tag{3}$$

where

$$\phi(u) = \begin{cases} \displaystyle\int_\Omega \left(\frac{1}{p}|\nabla u(x)|^p - \int_0^u \mathbf{b}_1(v)dv\right)dx, & u \in W_0^{1,p}(\Omega), \\ +\infty\ , & \text{otherwise.} \end{cases}$$

We will suppose throughout this section that **H-1, H-2, H-3, H-4** and **H-5** are satisfied. Thus, given $u_0 \in W_0^{1,p}(\Omega)$ there exists $u \in C(0, +\infty; L^2(\Omega))$ satisfying equation (3) a.e. in $(0, +\infty)$ such that $u(0) = u_0$. See [11] for details. We also impose conditions in $\mathbf{b}_1$ in such a way that $D(\phi) = D(\varphi) = W_0^{1,p}(\Omega)$. We only consider values of $p > n/2$. In this case, according to Theorem 4.4 (see Appendix), $D(\partial\phi) \subset L^\infty(\Omega)$.

3.1. Comparison

The aim of this subsection is to show that for large times the pseudo-semigroup $\{V_t\}$ behaves as a continuous semigroup $\{T_t\}$. To accomplish that the main tools are the comparison results obtained in [5].

The hypothesis **H-6** and **H-8** allow us to compare the solutions of equation (3) with those of equations presenting simple perturbations, by applying the abstract comparison results in [5]. For that we introduce in $H = L^2(\Omega)$ the order given by: if $u, v \in L^2(\Omega)$, then $u \geq v \Leftrightarrow u(x) \geq v(x)$ a.e. in Ω.

Now consider the following auxiliary equations:

$$\frac{d}{dt}v(t) + \partial\phi(v(t)) = kv(t) + k_0, \quad t > 0, \tag{4}$$

$$\frac{d}{dt}w(t) + \partial\phi(w(t)) = c_1 w(t) + c_2, \quad t > 0, \tag{5}$$

and we denote by $u(t, u_0)$, $v(t, v_0)$, and $w(t, w_0)$ the solutions of equations (3), (4), and (5), with initial dates u_0, v_0, and w_0, respectively. Let us first compare (3) and (4).

The following lemma is a particular case of Theorem 3.10, [3].

LEMMA 3.1. *If $v(t,v_0)$ is solution of (4) with $v(0,v_0)=v_0$, then*

$$\|v_t(t,v_0)\|_H \leq \frac{1}{t}\sup_K \|v_0-\eta\|_H, \quad \forall\, t>0,$$

where K is the set

$$K := \{\eta \in H : -\Delta_p \eta - B_1(\eta) = k\eta + k_0\}.$$

Note that K is a bounded subset of H and of $W_0^{1,p}(\Omega)$.

LEMMA 3.2. *With the above notation there exists a constant $\mathcal{C}(k,k_0)=\mathcal{C}$ depending only on k and k_0 such that, if $\mathcal{M}\subset W_0^{1,p}(\Omega)$ is a bounded subset, for some $t(\mathcal{M})>0$ we have*

$$u(t,u_0) \geq -\mathcal{C} \quad \forall\, t>t(\mathcal{M}), \qquad a.e.\ in\ \Omega, \tag{6}$$

for all $u_0\in\mathcal{M}$.

Proof: Since we are supposing $B_2(u)\geq ku+k_0\ \forall\, u\in H$, we have that $u_0\geq v_0$ implies $u(t,u_0)\geq v(t,v_0)\ \forall\, t>0$ (See [5]). We also have that $v(t)\in D(\partial\phi)\subset L^\infty(\Omega)$ for each $t>0$, and so, according to Theorem 4.4 in Appendix, there exists a constant $\mathcal{C}(k,k_0,\|v\|_H,\|v_t(t)\|_H)$ depending only on k,k_0, $\|v(t)\|_H$ and $\|v_t(t)\|_H$ such that

$$\|v(t)\|_{L^\infty(\Omega)} \leq \mathcal{C}(k,k_0,\|v\|_H,\|v_t(t)\|_H) \qquad \forall\, t>0.$$

It follows that,

$$u(t,u_0)\geq v(t,u_0)\geq -\mathcal{C}(k,k_0,\|v\|_H,\|v_t(t)\|_H) \quad \forall\, t>0, \qquad \text{a.e. in } \Omega. \tag{7}$$

Therefore, since $\mathcal{M}$ is a bounded subset of H, from Lemma 3.1, we can choose $t_1(\mathcal{M})>0$ large enough so that

$$\|v_t(t,u_0)\|_H\leq 1, \qquad \text{if } t>t_1(\mathcal{M}), \forall\, u_0\in\mathcal{M}.$$

From the results in [4], there is a constant $\rho>0$ not depending on $\mathcal{M}$ and $t_2(\mathcal{M})>0$ such that

$$\|v(t,u_0)\|_H\leq \rho, \qquad \text{if } t>t_2(\mathcal{M}), \forall\, u_0\in\mathcal{M}.$$

This means, by (7), that there exists $t(\mathcal{M})>0$ such that

$$u(t,u_0)\geq v(t,u_0)\geq -\mathcal{C} \qquad \forall\, t>t(\mathcal{M}), \forall\, u_0\in\mathcal{M}$$

where the constant $\mathcal{C}$ does not depend on the set $\mathcal{M}$. ∎

By using the above information and still using the same notation we are able to compare (3) and (5). By **H-8**, $B_2(u(x))\leq c_1u(x)+c_2(-\mathcal{C})$, $\forall\, u\in H$, since $u(x)\geq -\mathcal{C}$ is verified. We thus have

$$B_2(u(t,u_0))\leq c_1u(t,u_0)+c_2 \text{ a.e in } \Omega, \forall\, u_0\in\mathcal{M},\ \forall\, t\geq t(\mathcal{M}). \tag{8}$$

Consider that $t_0 := t(\mathcal{M})$. Then $w_0 \geq u(t_0, u_0)$ implies $w(t, w_0) \geq u(t + t_0, u_0)$, according to the results in [5].

LEMMA 3.3. *With the notation established above, if $\mathcal{M}$ is a bounded subset of $W_0^{1,p}(\Omega)$, then $U_{t_0} := \{u(t_0, u_0); u_0 \in \mathcal{M}\}$ is a bounded subset of $W_0^{1,p}(\Omega)$.*

Proof: We Multiply (3) by u, integrate over Ω and then integrate from 0 to t to obtain

$$\frac{1}{2}\|u(t)\|_H^2 + \int_0^t \phi(u(s))ds \leq \frac{1}{2}\|u_0\|_H^2 + \int_0^t \langle B_2 u(s), u(s)\rangle ds \leq \frac{1}{2}\|u_0\|_H^2 + \int_0^t \beta(\|u(s)\|_H^2 + 1)ds,$$

where we have used that $-B_1(u)u \geq -\int_0^u \mathbf{b}_1(s)ds$. By Gronwall's inequality

$$\|u(t)\|_H^2 \leq \mathcal{C}_0(\mathcal{M}, \beta, t_0), \quad t \leq t_0.$$

On the other hand we have that

$$\int_0^{t_0} \phi(u(s))ds \leq \frac{1}{2}\|u_0\|_H^2 + \beta \int_0^{t_0} (\|u(s)\|_H^2 + 1)ds,$$

and so

$$\int_0^{t_0} \phi(u(s))ds \leq \mathcal{C}_1(\mathcal{M}, \beta, t_0).$$

Since

$$\frac{d}{dt}\phi(u(t)) = \langle \partial\phi(u(t)), -\partial\phi(u(t))\rangle + \langle \partial\phi(u(t)), B_2 u(t)\rangle,$$

it follows that

$$\frac{1}{2}\int_0^t \|\partial\phi(u(s))\|_H^2 ds + \phi(u(t)) \leq \phi(u_0) + \frac{1}{2}\int_0^t \|B_2 u(s)\|_H^2 ds.$$

According with **H-4**,

$$\phi(u(t)) \leq \phi(u_0) + \frac{\mathcal{L}_0(\mathcal{C}_0(\mathcal{M}, \beta, t_0))}{2}\int_0^{t_0} [\{\phi(u(s))\}^2 + c]ds.$$

The result now follows applying Gronwall's inequality. ∎

Now we obtain bounds to $\|w(t, w_0)\|_{L^\infty(\Omega)}$. Again from Theorem 4.4 we have,

$$\|w(t, w_0)\|_{L^\infty(\Omega)} \leq \mathcal{C}(c_1, c_2, \|w(t, w_0)\|_H, \|w_t(t, w_0)\|_H), \tag{9}$$

which makes us look for bounds to $\|w\|_H$ and $\|w_t\|_H$. The next result follows from Lemma 4.11 in the Appendix.

LEMMA 3.4. *There are constants $\mathcal{K}_i = \mathcal{K}_i(c_1, p)$, $i = 0, 1, 2$, such that, given $\mathcal{M} \subset W_0^{1,p}(\Omega)$ bounded, there is $T_0 = T_0(\mathcal{M}) > 0$ satisfying*

$$\|w(t)\|_H \leq \mathcal{K}_0, \quad \|w(t)\|_{W_0^{1,p}(\Omega)} \leq \mathcal{K}_1 \ \ \text{and} \ \ \phi(w(t)) \leq \mathcal{K}_2, \quad \forall\, t \geq T_0(\mathcal{M}). \tag{10}$$

LEMMA 3.5. *There exists a constant $\tilde{\mathcal{K}} > 0$ such that, if $\mathcal{M}$ is a bounded subset of $W_0^{1,p}(\Omega)$ there exists $T_1 = T_1(\mathcal{M})$ which implies*

$$\|\frac{d}{dt}w(t, w_0)\|_H \leq \tilde{\mathcal{K}}, \quad t > T_1, \quad \forall w_0 \in \mathcal{M}.$$

Proof: If $0 \leq s < t$ and $h > 0$,

$$\frac{1}{2}\|w(t+h) - w(t)\|_H^2 \leq \frac{1}{2}\|w(s+h) - w(s)\|_H^2 + \int_s^t c_1\|w(\tau+h) - w(\tau)\|_H^2 d\tau.$$

Multiplying this equation by $1/h^2$ and letting $h \to 0$, we have

$$\|\frac{d}{dt}w(t)\|_H^2 \leq \|\frac{d}{dt}w(s)\|_H^2 + 2c_1\int_s^t \|\frac{d}{dt}w(\tau)\|_H^2 d\tau. \tag{11}$$

Let us first remark that

$$\begin{aligned}\frac{d}{dt}\phi(w(t)) &= \langle \partial\phi(w(t)), \frac{d}{dt}w(t)\rangle = \langle c_1 w(t) + c_2 - \frac{d}{dt}w(t), \frac{d}{dt}w(t)\rangle \\ &= \frac{1}{2c_1}\frac{d}{dt}\|c_1 w(t) + c_2\|_H^2 - \|\frac{d}{dt}w(t)\|_H^2.\end{aligned}$$

Therefore,

$$\int_s^t \|\frac{d}{dt}w(\tau)\|_H^2 d\tau + \phi(w(t)) \leq \phi(w(s)) + \frac{1}{2c_1}\|c_1 w(t) + c_2\|_H^2.$$

Choosing T_0 as in (10) and $T_0 \leq s < t$, then

$$\int_s^t \|\frac{d}{dt}w(\tau)\|_H^2 d\tau \leq \mathcal{K}_2 + \frac{1}{2c_1}\left(c_1\mathcal{K}_0 + c_2\right)^2 = \mathcal{K}. \tag{12}$$

It follows from (11) and (12) that

$$\|\frac{d}{dt}w(t)\|_H^2 \leq \|\frac{d}{dt}w(s)\|_H^2 + \mathcal{K}_3, \quad \text{where } \mathcal{K}_3 = 2c_1\mathcal{K}.$$

Integrating from T_0 to t, we obtain

$$\|\frac{d}{dt}w(t)\|_H^2 \leq \frac{1}{t-T_0}\int_{T_0}^{t}\|\frac{d}{dt}w(s)\|_H^2 ds + \mathcal{K}_3 \leq \mathcal{K}_3 + 1,$$

for $t \geq T_0 + \mathcal{K}$. ∎

According with above results we can conclude:

LEMMA 3.6. *There exists a constant $\tilde{\mathcal{C}}$ depending only on p, c_1, and c_2, such that if $\mathcal{M} \subset W_0^{1,p}(\Omega)$ is a bounded subset, then for some $T(\mathcal{M}) > 0$ we have*

$$u(t,u_0) \leq \tilde{\mathcal{C}} \quad \forall\, t > T(\mathcal{M}),\ \textit{a.e. in}\ \Omega, \quad \forall\, u_0 \in \mathcal{M}.$$

Proof: Consider $t_0 = t(\mathcal{M}) + 1$ where $t(\mathcal{M})$ is given in Lemma 3.2. If $\mathcal{M} \subset W_0^{1,p}(\Omega)$ is a bounded subset we define $U_{t_0}(\mathcal{M}) := \{u(t_0,u_0); u_0 \in \mathcal{M}\}$, then from Lemma 3.3, $U_{t_0}(\mathcal{M})$ is a bounded subset of $W_0^{1,p}(\Omega)$. It follows from the results in [5] that

$$w(t,u(t_0,u_0)) \geq u(t,u(t_0,u_0)), \quad \forall\, t > 0.$$

The proof now follows from (9) and Lemmas 3.4 and 3.5, with $T(\mathcal{M}) = \max\{t_0, T_1(\mathcal{M})\}$. ∎

Next we state the main result that we extract from the above lemmas:

THEOREM 3.1. *If $\mathcal{M} \subset W_0^{1,p}(\Omega)$ is a bounded subset, there exists $\tau_0 = \tau_0(\mathcal{M}) > 0$, and $\mathcal{N} = \mathcal{N}(k,k_0,c_1,c_2,p) > 0$, such that*

$$\|u(t,u_0)\|_{L^\infty(\Omega)} \leq \mathcal{N} \quad \forall\, t \geq \tau_0 \quad \forall\, u_0 \in \mathcal{M}. \tag{13}$$

Proof: In fact, with the same notation of Lemmas 3.2 and 3.6 and taking $t_0 = t(\mathcal{M})+1$, we have

$$-\mathcal{C}(k,k_0) \leq v(t,u_0) \leq u(t,u_0) \leq w(t-t_0,u(t_0,u_0)) \leq \tilde{\mathcal{C}}(c_1,c_2,p),$$

where $t \geq \tau_0 = \max\{t(\mathcal{M}), T(\mathcal{M})\}$. ∎

We are supposing in **H-9** that B_2 satisfies a local Lipschitz condition; that is, given a bounded set $\mathcal{M}$ in $W_0^{1,p}(\Omega)$, there exists $\omega = \omega(\mathcal{M})$, such that if u and $\bar{u}$ are solutions of (3), then

$$\|B_2(u(t,u_0)) - B_2(\bar{u}(t,\bar{u}_0))\|_H \leq \omega\|u(t,u_0) - \bar{u}(t,\bar{u}_0)\|_H, \quad t \geq \tau_0,\ u_0, \bar{u}_0 \in \mathcal{M}.$$

Therefore, if we are interested in large time behaviour, the solutions of equation (3) are essentially those of equations with simpler structure where the principal operator $-\Delta_p$ is perturbed by globally Lipschitz operators in H. In other words, denoting by $u(t,u_0)$ a solution of problem (3) starting in u_0, we have

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THEOREM 3.2. *Let us suppose that the Hypothesis* **H-6**, **H-7**, **H-8**, *and* **H-9** *are satisfied. There exists a globally Lipschitz application L in H such that if $\bar{u}(t, \bar{u}_0)$ is solution of*

$$\begin{cases} \dfrac{d}{dt}\bar{u}(t) + \partial\phi(\bar{u}(t)) = L(\bar{u}(t)), \quad t > 0, \\ \\ \bar{u}(0) = \bar{u}_0, \end{cases} \tag{14}$$

it follows that, if $\mathcal{M} \subset W_0^{1,p}(\Omega)$ is a bounded subset, there exists $\tau_0 = \tau_0(\mathcal{M}) > 0$ such that, if $\tau > \tau_0$ and $u_\tau = u(\tau, u_0), u_0 \in \mathcal{M}$, then $u(t, u_\tau) = \bar{u}(t, u_\tau)$ for all $t > 0$.

Proof: We can define $\ell : \mathbb{R} \to \mathbb{R}$ in the following way:

$$\ell(r) = \begin{cases} \mathbf{b}_2(r), & \text{if } |r| \le \mathcal{N}, \\ \mathbf{b}_2(-\mathcal{N}) & \text{if } r \le -\mathcal{N}, \\ \mathbf{b}_2(\mathcal{N}) & \text{if } r \ge \mathcal{N}, \end{cases}$$

where $\mathcal{N}$ is given in (13). Now, we only have to choose L as the Nemitskiĭ operator associated to ℓ. ∎

3.2. Existence of Attractors

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, $H = L^2(\Omega)$, and Δ_p the p-Laplacian operator with $p > \frac{n}{2}$, this restriction being made to guarantee that $D(\Delta_p) \subset L^\infty(\Omega)$. We will prove the existence of a global attractor for

$$\begin{cases} \dfrac{d}{dt}u(t) - \Delta_p u(t) + Bu(t) = 0, \quad t > 0, \\ \\ u(0) = u_0 \in W_0^{1,p}(\Omega), \end{cases} \tag{15}$$

where we assume that B can be decomposed in a sum of two other operators, i.e., $Bu = B_1 u + B_2 u$, $\forall\, u \in H$, with $-B_1$ and B_2 satisfying **H-6**, **H-7**, **H-8**, and **H-9**.

Next we rewrite equation (15) as

$$\frac{d}{dt}u(t) + \partial\phi(u(t)) = B_2(u(t)), \quad t > 0. \tag{16}$$

We also suppose that B_2 satisfies the Hypothesis **H-1, H-2, H-3, H-4, H-5**, in a way that, according to [11], for each initial data $u_0 \in W_0^{1,p}(\Omega)$ there exists at least one solution $u \in C(0, \infty; W_0^{1,p}(\Omega))$ of problem (16).

Assume that $\mathcal{X} = D(\phi) = W_0^{1,p}(\Omega) \subset\subset H$. We define in $\mathcal{X}$ the following operators family:

DEFINITION 3.1. Consider $x \in \mathcal{X}$. We denote by $S(x)$ the set

$$S(x) := \{u \in C(0, +\infty; H);\ u_t + \partial\phi(u) = B_2 u;\ u(0) = x\}.$$

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If $\mathcal{M} \subset \mathcal{X}$ we define

$$S(\mathcal{M}) := \bigcup_{x\in\mathcal{M}} S(x).$$

Let $\{V_t, t \in \mathbb{R}^+\}$ be a family of applications $\mathcal{X}$ in $\mathcal{P}(\mathcal{X})$, the power set of $\mathcal{X}$ subsets, given by

$$V_t(x) := \{u(t); u \in S(x)\}.$$

If $\mathcal{M} \subset \mathcal{X}$,

$$\begin{aligned} V_t(\mathcal{M}) &:= \{u(t); u \in S(\mathcal{M})\} = \{u(t); u \in \textstyle\bigcup_{x\in\mathcal{M}} S(x)\} \\ &= \textstyle\bigcup_{x\in\mathcal{M}}\{u(t); u \in S(x)\} = \bigcup_{x\in\mathcal{M}} V_t(x). \end{aligned}$$

$\{V_t\}$ satisfies:

$$V_0(x) = \{u(0); u \in S(x)\} = \{x\},$$

and we also have

$$\begin{aligned} V_{t_1+t_2}(x) &= \{u(t_1+t_2); u \in S(x)\} = \{u(t_1); u \in S\{u(t_2); u \in S(x)\}\} \\ &= \{u(t_1); u \in S(V_{t_2}(x))\} = V_{t_1}(V_{t_2}(x)), \end{aligned}$$

so $\{V_t\}$ is a pseudo-semigroup.

LEMMA 3.7. *Given a bounded subset $\mathcal{M} \subset W_0^{1,p}(\Omega)$, the ω-limit set of B, $\omega_V(\mathcal{M})$, is invariant under $\{V_t\}$.*

Proof: In fact, it is enough to verify that if $y \in \omega_V(\mathcal{M})$, then $\|y\|_{L^\infty(\Omega)} \leq \mathcal{N}$, where $\mathcal{N}$ is given in (13). This follows from Theorem 3.1 and from definition of $\omega -$ limit. ∎

THEOREM 3.3. *Under the above hypothesis and using the same notation, $\{V_t\}$ has a global attractor $\mathcal{A}$ in $W_0^{1,p}(\Omega)$, $\mathcal{A} \subset L^\infty(\Omega)$, and if $u \in \mathcal{A}$, $\|u\|_{L^\infty(\Omega)} \leq \mathcal{N}$, where $\mathcal{N}$ is given in (13).*

Proof: According to Theorem 3.2 there is a semigroup $\{T_t\}$ defined in $\mathcal{X}$ such that $\{V_t\}$ and $\{T_t\}$ satisfy Hypothesis **H-R**. The existence of an attractor follows from Theorem 2.2 and from Lemma 3.7. It follows as a corollary from proofs of Theorem 2.2 and Lemma 3.7 that

$$u \in \mathcal{A} \quad \Rightarrow \quad \|u\|_{L^\infty(\Omega)} \leq \mathcal{N}. \ ∎$$

3.3. Examples

EXAMPLE 3.1. We consider the problem

$$\begin{cases} \dfrac{d}{dt}u(t) - \Delta_p u(t) + |u(t)|^{r_1-2}u(t) = B_2(u(t)), \\ u(0) = u_0 \in W_0^{1,p}(\Omega), \end{cases} \tag{17}$$

where $2 \leq r_1 \leq np/(n-p)$ and B_2 is a Nemitskiĭ operator associated to a real and locally Lipschitz function $\mathbf{b}_2$, such that

[1] $|\mathbf{b}_2(s) - \mathbf{b}_2(\bar{s})| \leq c|s-\bar{s}|(|s|^{r_2-1} + |\bar{s}|^{r_2-1} + 1)$, with $r_2 < \frac{r_1}{2}$, $c \in \mathbb{R}^+$, $\forall\, s, \bar{s} \in \mathbb{R}$;

[2] $\limsup_{|s|\to\infty} \dfrac{\mathbf{b}_2(s)}{s} \leq \xi$ for some constant ξ;

[3] There are constants d_1 and d_2 such that $\mathbf{b}_2(r) \geq d_1 r + d_2$.

Under these conditions we can prove the existence of a global attractor in $W_0^{1,p}(\Omega)$ for the problem (17).

More generally, we have:

Example 3.2. Let B be a Nemitskiĭ operator associated to a real and continuous function b satisfying

$$\limsup_{s\to\infty} \frac{\mathbf{b}(s)}{s} \leq \xi$$

We define:

$$\mathbf{b}_1(r) := \inf_{s\leq r}[\mathbf{b}(s) - \xi s] - \mathbf{b}(0), \quad \text{and} \quad \mathbf{b}_2(r) := \mathbf{b}(r) - \mathbf{b}_1(r), \quad \forall\, r \in \mathbb{R}. \tag{18}$$

The functions $\mathbf{b}_1$ and $\mathbf{b}_2$ defined in this way are such that $\mathbf{b}_1(r)$ is decreasing, $\mathbf{b}_1(0) = 0$ and $\mathbf{b}_2(r) \geq \xi r + \mathbf{b}(0)$, $\forall\, r \in \mathbb{R}$. Let us also suppose

[1] $\mathbf{b}_1(s)s \geq c_0|s|^{r_0}$, with $r_0 \geq 2$, $c_0 \in \mathbb{R}^+$;

[2] $|\mathbf{b}_1(s)| \leq c_1(|s|^{r_1-1} + 1)$, with $r_1 \leq np/(n-p)$, $c_1 \in \mathbb{R}^+$;

[3] $|\mathbf{b}_2(s) - \mathbf{b}_2(\bar{s})| \leq c_2|s-\bar{s}|(|s|^{r_2-1} + |\bar{s}|^{r_2-1} + 1), \forall s, \bar{s} \in \mathbb{R}$, with $r_2 < \max\{\frac{r_0}{2}, p\}$, $c_2 \in \mathbb{R}^+$;

Let B_1 and B_2 be the Nemitskiĭ operators associated to $\mathbf{b}_1$ and $\mathbf{b}_2$, respectively. In such case $-\Delta_p - B_1$ is the subdifferential of:

$$\phi(u) = \begin{cases} \displaystyle\int_\Omega \left(\frac{1}{p}|\nabla u(x)|^p - \int_0^u \mathbf{b}_1(v)dv\right) dx, & u \in W_0^{1,p}(\Omega), \\ +\infty\,, & \text{otherwise.} \end{cases}$$

Under these conditions the problem

$$\begin{cases} u_t - \Delta_p u = Bu, \quad t > 0, \\ u(0) = u_0 \in W_0^{1,p}(\Omega), \end{cases} \tag{19}$$

has a global attractor in $W_0^{1,p}(\Omega)$.

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Remark 3. 2. Note that $\mathbf{b}_1$ is the monotone part of b and $\mathbf{b}_2$ is the oscillation part of b. Also note that the growth of $\mathbf{b}_2$ depends on how fast $\mathbf{b}_1$ grows (condition [1]). In fact we only need to make condition [1] be satisfied for large values of s since, if $\mathbf{b}_1$, defined as in (18), fails to satisfy condition [1] in a neighborhood of zero, we can take B_1 as the Nemitskiĭ operator associated to a suitable real function b_0 differing from $\mathbf{b}_1$ only in this neighborhood and satisfying the condition [1]. The difference between $\mathbf{b}_1$ and b_0 has compact support and we can add it to $\mathbf{b}_2$ without changing its properties.

3.4. Existence of Periodic Solutions

In this subsection we consider the problem

$$\begin{cases} u_t - \Delta_p u = B(u) + e(t) \\ \\ u(t_0) = u_0 \end{cases} \tag{20}$$

with homogeneous Dirichlet boundary conditions, where $e : \mathbb{R} \to \mathbb{R}$ is continuous and bounded function. Assume that B satisfies the hypotheses **H-1** thru **H-9**. Define $V_{t,t_0} : W_0^{1,p}(\Omega) \to \mathcal{P}(W_0^{1,p}(\Omega))$, $V_{t,t_0}(u_0) = \{u(t,t_0,u_0) : u(\cdot,t_0,u_0)$ is a solution of (20)$\}$. The family $V_{t,t_0} = \{V_{t,t_0}; t \geq t_0 \geq 0\}$ a *pseudo-evolution process*. For V_{t,t_0} we have that: Under these conditions, computations analogous to those of Subsection 3.1 V_{t,t_0} show that satisfies the **NH** condition. Computations analogous to those done in Subsection 4.1 show that the pseudo-evolution process V_{t,t_0} is bounded dissipative.

In the case when $e(t)$ is an $r-$periodic function the pseudo-evolution process V_{t,t_0} is $r-$periodic and we have, as a consequence of Theorem 2.3, that the discrete pseudo-semigroup associated to the operator $V_{r+\sigma,\sigma}$ has a global attractor and fixed point and therefore the problem (20) has a periodic solution.

4. APPENDIX

4.1. Attractors in $W_0^{1,p}(\Omega)$ for Lipschitz Perturbations of $-\Delta_p - B_1$

In [4] the authors prove the existence of a global attractors for the semigroup associated to (1) in $\overline{D(A)}$ when $B : H \to H$ is globally Lipschitz. In this section we consider the following particular case of (3): Assume that B_2 is a globally Lipschitz operator in $H = L^2(\Omega)$ and $\partial\phi u = (-\Delta_p - B_1)u = -\text{div}(|\nabla u|^{p-2}\nabla u) - \mathbf{b}_1(u)$ with homogeneous Dirichlet boundary condition.

Then, $\partial\phi$ is a maximal monotone operator, and $-\Delta_p - B_1$ where

$$\phi(u) = \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega |\nabla u(x)|^p dx - \int_\Omega \int_0^u \mathbf{b}_1(v)dv, & \text{if } u \in W_0^{1,p}(\Omega), \\ \\ +\infty \, , & \text{otherwise.} \end{cases}$$

ϕ is a convex, proper, l.s.c. application in H, $D(\phi) = W_0^{1,p}(\Omega)$, and $\phi(u) \geq \frac{1}{p}\|\cdot\|_{W_0^{1,p}(\Omega)}$. We prove the existence of a global attractor for (3) in $W_0^{1,p}(\Omega) = D(\phi) \supset D(\partial\phi)$. First we claim that if S_t is the semigroup in $H = L^2(\Omega)$ associated to problem (3), then $S_t(W_0^{1,p}(\Omega)) \subset W_0^{1,p}(\Omega)$.

THEOREM 4.1. *Let A be a maximal monotone operator and B globally Lipschitz in H with Lipschitz constant ω. If $u_0 \in D(A)$ and u is a weak solution of the problem*

$$\begin{cases} \dfrac{d}{dt}u(t) + Au(t) \ni Bu(t), \\ \\ u(0) = u_0, \end{cases} \tag{21}$$

then $u(t) \in D(A)$ for all $t \in [0,T]$.

Proof: In fact, since B is globally Lipschitz in H and $u \in C(0,T;H)$, it follows that $B(u) \in C(0,T;H)$. Then if $t \in [0,T]$, t is a Lebesgue's point of $Bu(t) = f(t)$. Since $u_0 \in D(A)$, from Theorem 3.5, [3], u is differentiable from right at $t = 0$. Thus we have

$$\lim_{h\downarrow 0} \frac{u(h) - u_0}{h} = \frac{du^+}{dt}u(0),$$

and so, given $\varepsilon > 0$ there is h small enough satisfying

$$\|u(h) - u_0\|_H \leq \varepsilon h + \left\| \frac{d}{dt}^+ u(0) \right\|_H h = kh.$$

Then for all $t_0 \in [0,T]$ we have

$$\begin{aligned} \|u(t_0 + h) - u(t_0)\|_H &\leq \|u(h) - u_0\|_H + \int_0^{t_0} \|B(u(s+h)) - B(u(s))\|_H ds \\ &\leq kh + \omega \int_0^{t_0} \|u(s+h) - u(s)\|_H ds. \end{aligned}$$

and this implies

$$\|u(t_0 + h) - u(t_0)\| \leq khe^{\omega t_0}.$$

Therefore

$$\liminf_{h\downarrow 0} \frac{1}{h}\|u(t_0 + h) - u(t_0)\| < \infty,$$

and again from Theorem 3.5, [3], $u(t_0) \in D(A)$. ∎

As a Corollary of the above Theorem we have

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COROLLARY 4.1. *If $u_0 \in H$ and u is a strong solution of (21) then $u(t) \in D(A)$ for all $t > 0$.*

Remark 4. 1. In the case $A = \partial\phi$, with ϕ convex, proper and l.s.c. we have that $\forall u_0 \in D(\phi)$, $u(t, u_0) \in D(\phi)$, $t \geq 0$.

Now we prove a lemma which enables us to conclude that the restriction of S_t to $W_0^{1,p}(\Omega)$ is a continuous and compact semigroup in this space.

LEMMA 4.8. *Let $u(t, u_0)$ denote the solution of (3) satisfying $u(0, u_0) = u_0$ where B_2 is assumed globally Lipschitz operator in H with Lipschitz constant ω. Let $\{u_{0n}\}$ be a bounded sequence in $W_0^{1,p}(\Omega)$ which converges in H to u_0. Then $u(t) := u(t, u_0) \in W_0^{1,p}(\Omega)$ and there exists a subsequence $u_{n_k}(t) := u(t, u_{0n_k})$ of $u_n(t) := u(t, u_{0n})$ converging to $u(t)$ in $W_0^{1,p}(\Omega)$ $\forall\, t > 0$.*

Proof: Multiplying the equation (3) by $u_n(t)$ we get

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_H^2 + \phi(u_n(t)) \leq \langle B_2 u_n(t), u_n(t)\rangle,$$

and so

$$\frac{1}{2}\|u_n(t)\|_H^2 + \int_0^t \phi(u_n(s))ds \leq \frac{1}{2}\|u_{0n}\|_H^2 + \int_0^t \omega(\|u_n(s)\|_H^2 + c\|u_n(s)\|_H)ds,$$

for some constant c and $t > 0$. Taking into account that there is a constant K such that $\|u_{0n}\|_H \leq K, \forall\, n \in \mathbb{N}$, it follows from Gronwall's inequality that, if $t \in [0, T]$ for some arbitrary $T > 0$, $\|u_n(t)\|_H \leq K_0(K, T, \omega)$. So there is a constant C_0 depending on K, ω and T, such that

$$\sup_{0\leq t\leq T}\left[\|u_n(t)\|_H + \int_0^t \phi(u_n(s))ds\right] \leq C_0. \tag{22}$$

Moreover

$$\frac{d}{dt}\phi(u_n(t)) = \langle \partial\phi(u_n(t)), \frac{d}{dt}u_n(t)\rangle = \langle \partial\phi(u_n(t)), \frac{d}{dt}u_n(t) - B_2 u_n(t)\rangle + \langle \partial\phi(u_n(t)), B_2 u_n(t)\rangle$$

which implies that

$$\begin{aligned}\int_0^t \|\partial\phi(u_n(s))\|_H^2 ds \;+\; \phi(u_n(t)) &\leq \phi(u_{0n}) + \int_0^t \|\partial\phi(u_n(s))\|_H \|B_2 u_n(s)\|_H ds \\ &\leq \int_0^t \frac{1}{2}\|\partial\phi(u_n(s))\|_H^2 ds + \int_0^t \frac{1}{2}\|B_2 u_n(s)\|_H^2 ds + \phi(u_{0n})\end{aligned}$$

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and so

$$\int_0^t \frac{1}{2}\,\|\partial\phi(u_n(s))\|_H^2 ds + \phi(u_n(t)) \le \phi(u_{0n}) + \int_0^t \frac{1}{2}\|B_2 u_n(s)\|_H^2 ds$$

$$\le\ \phi(u_{0n}) + \int_0^t \frac{\omega}{2}(\|(u_n(s))\|_H + c)^2 ds \le \phi(u_{0n}) + \tilde{c}\int_0^t [\phi(u_n(s)) + 1]^2 ds$$

From Gronwall's inequality there are constants K_1 and K_2, such that

$$\max_{\{0\le t\le T\}} \phi(u_n(t)) \le K_1, \qquad \int_0^t \|\partial\phi(u_n(s))\|_H^2 ds \le K_2, \qquad \forall\, n \in \mathbb{N}.$$

We know that $\forall\, t > 0$, $u_n(t) \to u(t)$ strongly in H, uniformly in [0,T], and $u(t) \in W_0^{1,p}(\Omega)$. By fixing $t > 0$, since $\|u_n(t)\|_{W_0^{1,p}(\Omega)} \le K_1$, there is $\{u_{nk}(t)\} \subset \{u_n(t)\}$, such that

$$u_{nk}(t) \rightharpoonup u(t) \qquad \text{weakly in } W_0^{1,p}(\Omega).$$

We now show that $\|u_{nk}(t)\|_{W_0^{1,p}(\Omega)} \to \|u(t)\|_{W_0^{1,p}(\Omega)}$. In fact there is $\delta > 0$ (see [6]), such that

$$\|u_{nk}(t) - u(t)\|^p_{W_0^{1,p}(\Omega)} \le \delta\langle \partial\phi(u_{nk}(t)) - \partial\phi(u(t)), u_{nk}(t) - u(t)\rangle,$$

so by Lebesgue Dominated Convergence Theorem

$$\int_0^T \|u_{nk}(s) - u(s)\|^p_{W_0^{1,p}(\Omega)} ds \le \delta \int_0^T \langle \partial\phi(u_{nk}(s)) - \partial\phi(u(s)), u_{nk}(s) - u(s)\rangle ds \to 0, \text{ as } n\to\infty.$$

From that we obtain $u_{nk} \to u$ in $L^p(0,T; W_0^{1,p}(\Omega))$ and, taking a subsequence $\{u_{nkl}(t)\}$, if necessary, we have that $\|u_{nkl}(t)\|_{W_0^{1,p}(\Omega)}$ converges to $\|u(t)\|_{W_0^{1,p}(\Omega)}$ a.e. in $[0,T]$. Given $t \in [0,T]$,

$$\begin{aligned}
|\phi(u_{nkl}(t)) - \phi(u(t))| &\le |\phi(u_{nkl}(t)) - \phi(u_{nkl}(\theta))| + |\phi(u_{nkl}(\theta)) - \phi(u(\theta))| \\
&+ |\phi(u(\theta)) - \phi(u(t))| \\
&\le \int_\theta^t |\langle \partial\phi(u_{nkl}(s)), \frac{d}{dt}u_{nkl}(s)\rangle| ds + |\phi(u_{nkl}(\theta)) - \phi(u(\theta))| \\
&+ \int_\theta^t |\langle \partial\phi(u(s)), \frac{d}{dt}u(s)\rangle| ds.
\end{aligned}$$

Choosing θ in a small enough neighborhood of t and such that $\phi(u_{nkl}(\theta)) \to \phi(u(\theta))$, we can make $|\phi(u_{nkl}(t)) - \phi(u(t))|$, as small as we wish since

$$\langle \partial\phi(u_{nkl}(s)), \frac{d}{dt}u_{nkl}(s)\rangle \le \frac{1}{2}\|B_2 u_{nkl}(s)\|_H^2,$$

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and the right hand side of the above inequality is uniformly bounded.

Since t is arbitrary we can conclude that $\phi(u_{nkl}(t)) \to \phi(u(t))$ for all $t \in [0,T]$, for any $T > 0$. The result follows observing that (see Proposition 1.4, [2])

$$u_{nkl}(t) \overset{W_0^{1,p}(\Omega)}{\rightharpoonup} u(t) \text{ and } \|u_{nkl}(t)\|_{W_0^{1,p}(\Omega)} \to \|u(t)\|_{W_0^{1,p}(\Omega)},$$

imply

$$u_{nkl}(t) \overset{W_0^{1,p}(\Omega)}{\to} u(t). \blacksquare$$

The following lemma ensures that the semigroup defined by (3) in $W_0^{1,p}(\Omega)$ is compact.

LEMMA 4.9. *With the above notation, if $\mathcal{M} \subset W_0^{1,p}(\Omega)$ is a bounded subset, there is $T_0 > 0$ such that if $t > T_0$ and $\{u_n(t)\} \subset \{u_n(t,\mathcal{M})\}$, then $\{u_n(t)\}$ has a converging subsequence.*

LEMMA 4.10. *Let $u(t) \in C(0,+\infty; L^2(\Omega))$ be a solution of problem (1). The following applications are continuous:*

- (i) $\mathbb{R}^+ \ni t \mapsto u(t) \in W_0^{1,p}(\Omega)$;
- (ii) $W_0^{1,p}(\Omega) \ni u_0 \mapsto u(t) \in W_0^{1,p}(\Omega)$.

Let us denote by S_t the restriction to $W_0^{1,p}(\Omega)$ of the semigroup associated to problem (1). The compactness of S_t in $W_0^{1,p}(\Omega)$ has already been established in Lemma 4.9. In order to obtain the dissipative property of S_t in $W_0^{1,p}(\Omega)$ we use the Uniform Gronwall's Lemma (see [13], Lemma 1.1, page 89).

LEMMA 4.11. *The semigroup S_t associated to problem (1), defined in $W_0^{1,p}(\Omega)$, is a bounded dissipative semigroup.*

Proof: Multiplying the equation (3) by $\frac{d}{dt}u(t)$ and using the Young's inequality we get

$$\frac{1}{2}\|\frac{d}{dt}u(t)\|_H^2 + \frac{d}{dt}\phi(u(t)) \le \frac{1}{2}\|B_2u(t)\|_H^2.$$

Since B_2 is assumed globally Lipschitz in H and $\phi(u) \ge \frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)}$, there are constants c_1 and c_2 such that

$$\frac{d}{dt}\phi(u(t)) \le c_1[\phi(u(t))]^2 + c_2.$$

We note that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_H^2 + \phi(u(t)) \le \frac{1}{2}\frac{d}{dt}\|u(t)\|_H^2 + \langle\partial\phi(u(t)), u(t)\rangle = \langle B_2u(t), u(t)\rangle,$$

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hence

$$\frac{1}{2}\|u(t+r)\|_H^2 + \int_t^{t+r} \phi(u(s))ds \leq \frac{1}{2}\|u(t)\|_H^2 + \int_t^{t+r} \|B_2 u(s)\|_H \|u(s)\|_H ds.$$

There is a constant $k > 0$ which does not depend on u_0 such that for a large enough t, $\|u(t)\|_H \leq k$ (see [4]). Therefore

$$\int_t^{t+r} \phi(u(s))ds \leq \mathcal{C}(k, r, \omega),$$

where ω is the Lipschitz constant of B_2. From the Uniform Gronwall's Lemma there is a constant $\tilde{\mathcal{K}}$ depending only on k, r, and ω such that if t_0 is large enough,

$$\phi(u(t+r)) \leq \tilde{\mathcal{K}}, \qquad \forall\, t \geq t_0. \blacksquare$$

Thus, from Theorem 3.4.6 in [7] and from Lemmas 4.9, 4.10, and 4.11 we obtain the following result:

Theorem 4.2. *Let $\{S_t\}$ be the restriction to $W_0^{1,p}(\Omega)$ of the semigroup $\{T_t\}$ associated to problem (3). $\{S_t\}$ has a global attractor in $W_0^{1,p}(\Omega)$.*

4.2. Domain Regularity

In this appendix we partially generalize the Elliptic Lemma [1]. In order to obtain information about the domain of p-Laplacian operator, we consider the problem:

$$\begin{cases} -\text{div}(a|\nabla u|^{p-2}\nabla u) + g(u) = f(x) & \text{in } \Omega, \\ u = 0 \quad \text{in } \partial\Omega, \end{cases} \tag{23}$$

where $a \in L^\infty(\Omega)$, $\langle g(u), u\rangle \geq 0$ for all $u \in D(\Delta_p)$, and $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. Since the solution of (23) will be in $W_0^{1,p}(\Omega) \subset L^\infty(\Omega)$ for $p > n$, we are interested only in the case where $p \leq n$. (The case $p = n$ is treated in a similar way, so we only consider the case $2 \leq p < n$.)

Theorem 4.3. *If $f \in L^{1+1/p+1/n}(\Omega)$, then the solution u of (23) belongs to $L^{np/(n-p)}(\Omega)$ and*

$$\|u\|_{L^{np/(n-p)}(\Omega)} \leq \|f\|^{1/(p-1)}_{L^{1+1/p+1/n}(\Omega)}.$$

Proof: Multiplying the equation (23) by u and integrating it by parts we get

$$\int_\Omega a|\nabla u|^p \leq \int_\Omega fu \leq \|f\|_{L^{1+1/p+1/n}(\Omega)}\|u\|_{L^{np/(n-p)}(\Omega)},$$

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and since $\int_\Omega a|\nabla u|^p dx \geq m\int_\Omega |\nabla u|^p dx \geq cm\|u\|^p_{L^{np/(n-p)}(\Omega)}$ we have

$$cm\|u\|^{p-1}_{L^{np/(n-p)}(\Omega)} \leq \|f\|_{L^{1+1/p+1/n}(\Omega)}$$

and the result follows. ∎

Theorem 4.4. *If $f \in L^\nu(\Omega)$ with $\nu > n/p$ then $u \in L^\infty(\Omega)$ and*

$$\|u\|_{L^\infty(\Omega)} \leq C(\|f\|_{L^\nu(\Omega)}, inf\{a\}). \tag{24}$$

Proof: Consider $\varphi = (u-k)^+ = \max\{u(x)-k, 0\}$ and $A_k = \{x \in \Omega : u(x) > k\}$. If we multiply equation (23) by φ and integrate it by parts, we have

$$\int_\Omega a|\nabla\varphi|^p dx \leq \int_\Omega f\varphi dx \leq \|f\|_{L^\nu(\Omega)}\|\varphi\|_{L^{\nu'}(\Omega)} \leq \|f\|_{L^\nu(\Omega)}\|\varphi\|_{L^{np/(n-p)}(\Omega)}|A_k|^{1/\nu'-(n-p)/np}.$$

Note that

$$\int_\Omega a|\nabla\varphi|^p dx \geq m\left(\int_\Omega |\nabla\varphi|^p dx\right)^{\frac{p-1}{p}}\left(\int_\Omega |\nabla\varphi|^p dx\right)^{\frac{1}{p}} \geq cm\left(\int_\Omega |\nabla\varphi|^p dx\right)^{\frac{p-1}{p}}\|\varphi\|_{L^{\frac{np}{n-p}}(\Omega)}.$$

Therefore,

$$cm\left(\int_\Omega |\nabla\varphi|^p dx\right)^{(p-1)/p} \leq \|f\|_{L^\nu(\Omega)}|A_k|^{1/\nu'-(n-p)/np}.$$

Moreover

$$\|\varphi\|_{L^1(\Omega)}|A_k|^{-1+1/p-1/n} \leq c\|\varphi\|_{L^{np/(n-p)}(\Omega)}$$

and

$$(\|\varphi\|_{L^1(\Omega)}|A_k|^{-1+1/p-1/n})^{p-1} \leq c\|f\|_{L^\nu(\Omega)}|A_k|^{1/\nu'-(n-p)/np}.$$

Thus

$$\|\varphi\|_{L^1(\Omega)} \leq c\|f\|^{1/p-1}_{L^\nu(\Omega)}|A_k|^{\frac{1}{\nu'(p-1)}-\frac{n-p}{np(p-1)}+1-\frac{1}{p}+\frac{1}{n}}.$$

It follows that

$$\|\varphi\|_{L^1(\Omega)} \leq c\|f\|^{1/(p-1)}_{L^\nu(\Omega)}|A_k|^{1+\frac{1}{(p-1)}(\frac{1}{\nu'}-\frac{n-p}{n})}$$

and $\frac{1}{\nu'} - \frac{n-p}{n} = \frac{p}{n} - \frac{1}{\nu} > 0$ since $\nu > n/p$. The result follows from Lemma 5.1 [9] . ∎

REFERENCES

1. Arrieta, J.; Carvalho, A.N. Rodriguez-Bernal, A., Attractors of Parabolic Problems with Nonlinear Boundary Condition. Uniform Bounds. *Communications in Partial Differential Equations* to appear.

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Sob a supervisão CPq/ICMC

2. Barbu, V., *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff International Publishing, (1976).
3. Brèzis, H., *Operateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert*, North-Holland Publishing Company, Amsterdam, (1973).
4. Carvalho, A.N.; Cholewa, J.W. and Dlotko, T. Global Attractors for Problems with Monotone Operators, *Bolletino della Unione Matematica Italiana* to appear.
5. Carvalho, A. N. and Gentile C. B., Comparison Results for Nonlinear Parabolic Problems with Monotone Principal Part. Preprint
6. DiBennedeto, E., *Degenerate Parabolic Equations*, Springer-Verlag, New York, (1993).
7. Hale, J.K., *Asymptotic Behaviour of Dissipative Systems*, Mathematical Surveys and Monographs, **25**, American Mathematical Society, (1989).
8. Ladyzhenskaya, O., *Attractors for Semigroups and Evolution Equations*, Cambridge university Press, Cambridge, (1991).
9. Ladyzhenskaya, O., Ural'tseva, N., *Linear and Quasilinear Elliptic Equations*, Academic Press, London, (1968).
10. Lions, J.L., *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*, Dunod, Paris, (1969).
11. Ôtani, M., Nonmonotone Perturbations for Nonlinear Parabolic Equations Associated Whit Subdifferential Operators, Cauchy Problems, *Journal of Differential Equations* 46, 268-299 ,1982.
12. Pazy, A., *Semi-Groups of Nonlinear Contractions and Their Asymptotic Behaviour*, Nonlinear Analysis and mechanics: Heriot-watt Symposium, Vol. III, Research Notes in Mathematics, Pitman Publishing Limited, London, 1979.
13. Temam, R., *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, New York, 1988.
14. Vrabie, I.I., *Compactness Methods for Nonlinear Evolutions*, Pitman Monographs and Surveys in Pure and Applied Mathematics, London, 1987.

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