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A Formula for the Area of a Surface

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In this paper we present a formula for the computation of the area of a surface, which is a kind of curvilinear Fubini's formula (Theorem 2). Such formula relates the area of a surface with the integral of the *period map* of an ordinary differential equation on that surface. Some applications are given.
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1. THE MAIN RESULT

Let M be an oriented 2-dimensional Riemannian manifold of class C^1. Let $f\colon M \to \mathbb{R}$ be a C^1 function whose derivative at a point x is denoted by $f'(x)\colon T_xM \to \mathbb{R}$, where T_xM is the tangent space of M at x. There exists a unique vector $g(x) \in T_xM$ such that $f'(x)\cdot v = \langle g(x), v\rangle$, for all $v \in T_xM$, where $\langle\, ,\, \rangle$ denotes the inner product in T_xM. Let $u(x)$ be the vector field obtained by rotating $g(x)$ by an angle of $\frac{\pi}{2}$ radians. Observe that the vectors $u(x)$ are tangent to the level curves of f, so the vectors $g(x)$ are orthogonal to the level curves of f. Denote this rotation by Φ; then $u(x) = \Phi(g(x))$ and $\Phi\colon T_xM \to T_xM$ is a linear operator such that: (i) $\Phi^2 = -I$, (ii) $\langle v, \Phi v\rangle = 0$, (iii) $\|\Phi(v)\| = \|v\|$, (iv) $\{v, \Phi(v)\}$ is a positive basis, if $v \neq 0$.

Consider the differential equation

$$\dot{x} = u(x)\,. \tag{1}$$

For every regular value $y \in f(M)$ the set $f^{-1}(y)$ is invariant by Eq. (1), that is, each solution of Eq. (1) with initial condition in $f^{-1}(y)$ is contained in M and describes a piece of the level curve $f^{-1}(y)$. We also observe that we have uniqueness of solutions for the initial value problem

$$\begin{aligned} &\dot{x} = u(x) \\ &x(0) = x_0\,. \end{aligned} \tag{2}$$

In fact, a local solution of (2) is contained in the level curve $f(x) = c$, ($c = f(x_0)$) such that $\nabla f(x_0) \neq 0$. Suppose x, y are two solutions of (2). Since $\dot{x}(0) = u(x_0) = \dot{y}(0)$,

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the maps $x(t)$ and $y(t)$ are local parameterizations of the level curve $f^{-1}(c)$ through the point x_0. Thus there is a change of variable of class C^1, $s = h(t)$, $h(0) = c$, such that $x(t) = y(s)$. By the chain rule, $\dot{x}(t) = \dot{h}(t)\,\dot{y}(h(t))$, and therefore $u(x(t)) = \dot{h}(t)\,u(y(h(t))) = \dot{h}(t)\,u(x(t))$, which implies $\dot{h}(t) = 1$, and hence $h(t) = t$, for all t. This shows the uniqueness of solutions of (2).

Definition: We define the *period map* $p\colon \mathbb{R} \to [0,\infty) \cup \{+\infty\}$ of (1) in the following way: for each regular value $y \in f(M)$, there exists a finite or countable union of disjoint intervals $J_y = \cup_\lambda J_\lambda \subset \mathbb{R}$ and a solution $x\colon J_y \to M$ of (1) (since the set J_y may be disconnected, we mean solution in an extended sense: the orbit is a union of orbits in the usual sense) such that x is $1-1$ and $x(J_y) = f^{-1}(y)$. Then we define $p(y) = \sum_\lambda l(J_\lambda)$, where $l(J_\lambda)$ is the length of J_λ. For each $y \in f(M)$ that is not a regular value of f, we put $p(y) = +\infty$; and if $f^{-1}(y) = \emptyset$, we define $p(y) = 0$.

We remark that the period function at a point z corresponds to the time elapsed by the solution to traverse the orbit $f^{-1}(z)$. When this orbit is periodic, $p(z)$ is the minimal period of the orbit. We recall that an extended real valued function q is said to be *lower semi continuous* at a point z_0 such that $q(z_0) < \infty$ if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $q(z_0) - \varepsilon < q(z)$, whenever $|z - z_0| < \delta$. It is lower semi continuous at a z_0 such that $q(z_0) = \infty$ if, for any L, there is a $\delta > 0$ such that $q(z) > L$, whenever $|z - z_0| < \delta$.

Theorem 1.1. *The period function p is lower semi continuous.*

Proof: We shall only present the proof for the case $p(z_0) < \infty$. Since $p(z_0) < \infty$, the series $\sum_\lambda l(J_\lambda)$ is convergent. Given $\varepsilon > 0$, there exists n such that $\sum_{i=n+1}^{\infty} l(J_i) < \varepsilon$. Using the theorem on local form of submersions (see [7]) and the continuity of solutions of differential equations with respect to initial conditions we can conclude the existence of a finite family of closed intervals $\{Q_0\,, Q_1\,, \ldots, Q_n\}$, $Q_i \subset J_i\ \forall i$, satisfying $l(J_i) - l(Q_i) < \varepsilon/n$, and exists $\bar{\delta} > 0$ such that is $|x_0 - \bar{x}_0| < \bar{\delta}$, the solution $x(t, \bar{x}_0)$ is defined at least on Q_i. Hence

$$p\bar{z}_0 \geq \sum_{i=1}^{n} l(Q_i) > \sum_{i=1}^{n} [l(J_i) - \varepsilon] > [p(z_0) - \varepsilon] - \varepsilon = p(z_0) - 2\varepsilon.$$

Remark 1: Let $p\colon \mathbb{R} \to [0,\infty) \cup \{+\infty\}$ be a given lower semi continuous function and $M = \{(t,z) \in \mathbb{R}^2, 0 < t < p(z)\}$. Since M is open, then M is a Riemannian manifold with the induced metric of $\mathbb{R}^2$). If $f\colon M \to \mathbb{R}$ is given by $f(t,z) = z$, the correspondent period map for this function f is p.

Remark 2: The period map is Lebesgue measurable, but may not be Riemann integrable, even if $f(M)$ and p are bounded. Let $A \subset (0,1)$ be a closed set with positive measure and empty interior. Consider the open set $M = ((0,1) - A) \times (-2,2) \cup (A \times (-1,1))$ and the map $p(x)$ defined by $p(x) = 2$ if $x \in A$ and $p(x) = 4$ if $x \notin A$. The function p is discontinuous in A and since the measure of A is not zero, then p is not Riemann integrable.

In the proof of Lemma 2 below we need the following equality, related to Schwarz inequality, whose proof is immediate.

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LEMMA 1.1. *Let u, v vectors in a 2-dimensional Hilbert space, and let $\tilde{u}$ be a vector that is orthogonal to u such that $\|\tilde{u}\| = \|u\|$. Then*

$$\|u\|^2\|v\|^2 = \langle\, u, v\,\rangle^2 + \langle\, \tilde{u}, v\,\rangle^2.$$

LEMMA 1.2. *Let M be a 2-dimensional oriented Riemannian manifold with boundary, and let $f\colon M \to \mathbb{R}$ be a C^1 map such that:*
(a) f and $f|_{\partial M}$ have no critical points;
(b) For any $z \in f(M)$, there exists an interval $[r, s]$, $r < z < s$, such that $f^{-1}[r, s]$ is compact in M;
(c) For any $z \in f(M)$, the set $D(z) = \{x \in M \,: f(x) < z\}$ has finite area $A(z)$.
For each $x \in M$, let $\tilde{A}(x) = A(f(x))$ and $\tilde{p}(x) = p\,(f(x))$. Then:
(i) $A'(z) = p(z), \quad \forall z \in f(M)$
(ii) $\tilde{A}'(x) = \tilde{p}(x)\, f'(x), \quad \forall x \in M$.

Proof: Since we can work in each connected component of $f^{-1}(z)$, we shall assume that $f^{-1}(z)$ is connected, for all z. Let $z \in J$, then $f^{-1}(z)$ is an orbit which we will parameterize by z. For any x_0 let $z_0 = f(x_0)$ and consider the initial value problem

$$\varphi'(z) = \frac{g(\varphi(z))}{\|g(\varphi(z))\|^2}$$
$$\varphi(z_0) = x_0.$$

Since we work in each chart of M we can assume that there exists a z_0 such that the orbit of the corresponding solution $\varphi(z)$ intercepts every level curve of f, that is, the function $\varphi(z)$ is defined for all $z \in J$. Then we have

$$\langle\, g(\varphi(z))\,, \varphi'(z)\,\rangle = 1.$$

Integrating both sides, we get $f(\varphi(z)) = z + c$.
Since $f(\varphi(z_0)) = f(x_0) = z_0$, we have $z_0 = f(x_0) = f(\varphi(z_0)) = z_0 + c$, which implies $c = 0$. Therefore $f(\varphi(z)) = z$, $\forall z \in J$. Now consider the solutions $x(t, z)$ of the differential equation $\dot{x} = u(x)$ such that $x(0, z) = \varphi(z)$; then clearly $f(x(t, z)) = z, \forall z$. The map $x(t, z)$ parameterizes M; If $W = \{(t, z) \,:\, 0 < t < p(z),\, z \in J\}$ then $x\colon W \to M$ is a diffeomorphism.We claim that if $E = \|\frac{\partial x}{\partial t}\|^2$, $G = \|\frac{\partial x}{\partial z}\|^2, F = \langle \frac{\partial x}{\partial t}, \frac{\partial x}{\partial z}\rangle$, then $EG - F^2 = 1$, for all $t\,, z$. In fact, from $f(\varphi(z)) = z$, $\forall z \in J$, is follows that $f(x(t, z)) = z$, $\forall z \in J$, for $x(0, z) = \varphi(x)$ and $x(t, z)$ is a level curve of f. Note that since $\Phi^2 = -I$, $\frac{\partial x}{\partial t} = -\Phi\frac{\partial x}{\partial z}$ implies $-\Phi\big(\frac{\partial x}{\partial z}\big) = g$. Hence $1 = \langle\, g(x)\,, \frac{\partial x}{\partial z}\,\rangle = \langle -\Phi\big(\frac{\partial x}{\partial t}\big), \frac{\partial x}{\partial z}\rangle$ and it follows from Lemma 1 that

$$1 = \Big\langle\, g(x)\,, \frac{\partial x}{\partial z}\,\Big\rangle^2 = \Big\langle \Phi\Big(\frac{\partial x}{\partial z}\Big), \frac{\partial x}{\partial t}\Big\rangle^2 = \Big\|\frac{\partial x}{\partial t}\Big\|^2 \Big\|\frac{\partial x}{\partial z}\Big\|^2 - \Big\langle\frac{\partial x}{\partial t}, \frac{\partial x}{\partial t}\Big\rangle^2$$
$$= EG - F^2.$$

From the Differential Geometry we have

$$\text{Area}\,(M) = \int_W \sqrt{EG - F^2}\,dt\,dz = \int_W dt\,dz = \int_{z_0}^{z_1} p(z)\,dz.$$

Therefore $A(z) = A(z_0) + \int_{z_0}^{z} p(\tilde{z})d\tilde{z}$, and then $A'(z) = p(z)$. This shows (i). The proof of (ii) follows from (i) and the chain rule.

Lemma 1.3. *Let M be a compact oriented Riemannian manifold with boundary, and f be a C^1 map with no critical points. Then*

$$\text{Area}\,(M) = \int p(z)\,dz.$$

Proof: Denote by CV the set of critical values of $f|_{\partial M}$. Since M is compact, CV is a closed set. Since f has no critical points, Sard's Theorem implies $m(CV) = 0$. Then $\mathbb{R} \setminus CV$ is a countable union (possibly finite) of open intervals, that is, $\mathbb{R} \setminus CV = \cup I_n$, where I_n is an open interval for all n. Since the result holds in each I_n, Lemma 2 implies

$$\text{Area}\,(f^{-1}(I_n)) = \int_{I_n} p(x)\,dx.$$

Then, since the area of $f^{-1}(CV)$ is zero (because f has no critical points), we have

$$\begin{aligned}\int_{\mathbb{R}} p(z)\,dz \; &= \int_{\mathbb{R}\setminus CV} p(z)\,dz = \sum_n \int_{I_n} p(z)\,dz \\ &= \sum_n \text{Area}\,(f^{-1}(I_n)) = \text{Area}\,(M).\end{aligned}$$

Theorem 1.2. *Let M be an oriented Riemannian manifold without boundary, and let f be a C^1 map without critical points. Then*

$$\text{Area}\,(M) = \int p(z)\,dz.$$

Proof: We can write $M = \cup M_n$, a countable union, in which $M_n \subset \overset{\circ}{M}_{n+1}$ and each M_n is a compact manifold. Let the p_n be the period map in M_n. By Lemma 3, we have

$$\text{Area}\,(M_n) = \int p_n(z)\,dz.$$

Note that $p(z) = \lim_{n\to\infty} p_n(z)$, the sequence $(p_n(z))$ is bounded by $p(z)$ and $p_{n+1}(z) \geq p_n(z)$. By Lebesgue Dominated Convergence Theorem, we have

$$\text{Area}\,(M) = \lim_{n\to\infty} \text{Area}\,(M_n) = \lim_{n\to\infty} \int p_n(z)\,dz = \int p(z)\,dz.$$

Remark 3: Since each p_n is continuous and p is not Riemann integrable, the convergence cannot be uniform.

If the set of critical points of f has positive measure, we have the following result, whose proof is immediate.

COROLLARY 1.1. *Let M be an oriented Riemannian manifold without boundary, and let f be a C^1 map. Then*

$$\text{Area}\,(M) = \int p(z)\,dz + \text{Area}\,(C_f),$$

where C_f is the set of critical points of f.

Remark 4: If the differential equation (1) is $\dot{x} = \lambda(x)u(x), \quad \lambda(x) \neq 0$, then, $A'(z) = \int_0^{\tilde{p}(z)} \lambda(x(t,\bar{z}))dt$, where $\tilde{p}(z)$ is the period function for the map $\lambda(x)u(x)$. In fact, we have $1 = \langle\, g(x)\,, \dfrac{\partial x}{\partial z}\,\rangle = \langle\, -\Phi(u(x))\,, \dfrac{\partial x}{\partial t}\,\rangle = \dfrac{1}{\lambda(x)}\langle -\Phi\Big(\dfrac{\partial x}{\partial t}\Big), \dfrac{\partial x}{\partial z}\rangle.$

Since $\big\|\Phi(\frac{\partial x}{\partial t})\big\| = \big\|\frac{\partial x}{\partial t}\big\|$ and $\Phi(\frac{\partial x}{\partial t})$ is orthogonal to $\frac{\partial x}{\partial t}$, we have:

$$\Big\|\frac{\partial x}{\partial t}\Big\|^2\Big\|\frac{\partial x}{\partial z}\Big\|^2 = \langle\, \tfrac{\partial x}{\partial t},\ \tfrac{\partial x}{\partial z}\rangle^2 + \langle\, \Phi(\tfrac{\partial x}{\partial t}),\ \tfrac{\partial x}{\partial z}\,\rangle^2 = \langle\, \tfrac{\partial x}{\partial t},\ \tfrac{\partial x}{\partial z}\,\rangle^2 + (\lambda(x))^2.$$

Then, $EG - F^2 = \lambda^2(x)$; hence $A(z) - A(z_0) = \int_{z_0}^{z} d\bar{z} \int_0^{\tilde{p}(z)} \lambda(x(t,\bar{z}))dt$. Therefore, $A'(z) = \int_0^{\tilde{p}(z)} \lambda(x(t,\bar{z}))dt$.

2. EXAMPLES

1) Consider the cone $M = \{(x,y,z) \in \mathbb{R}^3 : x^2 = y^2 + z^2\}$ and the function $f\colon M \to \mathbb{R}$ defined by $f(x,y,z) = x$. Consider the zero level curve of f given by $\phi(x,y,z) = x^2 - y^2 - z^2 = 0$ and let $P = (x,y,z)$ be a point of M. The tangent plane of M at P is $T_PM = \{v = (v_1,v_2,v_3) : \langle\, \nabla\phi(x,y,z),\, v\,\rangle = 2xv_1 - 2yv_2 - 2zv_3 = 0\}$. For $v \in T_PM$ we have $f'(P).v = \langle\, \nabla f(P), v\rangle = \langle(1,0,0),(v_1,v_2,v_3)\rangle = v_1$. A vector $g(P) \in T_PM$ so that $\langle g(P), v\rangle = f'(P).v = v_1$ is

$$g(P) = \Big(\frac{y^2+z^2}{x^2+y^2+z^2}, \frac{y\,x}{x^2+y^2+z^2}, \frac{z\,x}{x^2+y^2+z^2}\Big).$$

The vector $u(P)$ is orthogonal to both $\nabla\phi(P)$ and $g(P)$, and $|u(P)| = |g(P)|$. Then $u(P) = \big(0, \frac{z}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}\big)$.

The differential equation is

$$\begin{aligned} &\dot{P} = u(P) \\ &P(0) = (x_0,y_0,z_0) \in M\,. \end{aligned} \tag{3}$$

The solution of (3) is $(x(t),y(t),z(t)) = (x_0, x_0\cos(\frac{1}{x_0\sqrt{2}}t),\, x_0\sin(\frac{1}{x_0\sqrt{2}}t))$.

Therefore, the period function is $p(x) = 2\sqrt{2}\pi x$ and hence the area of M is

$$A = 2\sqrt{2}\int_0^h \pi x dx = \sqrt{2}\pi h^2,$$

which coincides with the known formula for the area of the cone.

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2) As another example, consider the nonlinear second order scalar differential equation

$$\ddot{z} + 2z^3 = 0. \tag{4}$$

It can be written as a 2-dimensional differential equation

$$\dot{x} = u(x), \tag{5}$$

where $x = (x_1\,, x_2)$, $u(x_1\,, x_2) = (x_2\,, -2x_1^3)$. The orbits of (5) in the $(x_1\,, x_2)-$ plane are the level curves of the *total energy* function $f(x, y) = (x_2^2 + x_1^4)/2$. For each $h > 0$, the level curve

$$x_1^4 + x_2^2 = 2h\,, \tag{6}$$

is a periodic orbit of (5) in the phase plane $(x_1\,, x_2)$ that intersects the x_1-axis at the points $(-a, 0)$ and $(a, 0)$, where $a^4 = 2h$. Due to the symmetry of (6), this orbit is symmetric with respect to both x_1- and x_2- axes. Each corresponding solution has maximum amplitude a. By solving Eq. (6) for $x_2 = \frac{dx_1}{dt}$ and then integrating from 0 to a over a quarter of orbit we get the period $p(h)$ of this orbit as $p(h) = \frac{C}{(2h)^{1/4}}$, where $C = 4\int_0^1 \frac{du}{\sqrt{1-u^4}} = \frac{\sqrt{\pi}\Gamma(1/4)}{\Gamma(3/4)} \simeq 1.31103$. Note that $p(h) \to \infty$ as $h \to 0^+$, and $p(h) \to 0$ as $h \to \infty$. Then Theorem 2 implies that the area A of the annulus between the orbits corresponding to the energy levels h and k, $0 \le k < h$, is given by

$$A = \int_k^h p(\lambda)\, d\lambda = C \int_k^h \lambda^{-1/4}\, d\lambda = \frac{4C}{3}(h^{3/4} - k^{3/4}).$$

In particular, the area enclosed by the orbit corresponding to the energy level $h = a^4/2$ is

$$A = \frac{4C}{3} h^{3/4} = \frac{4C}{3\sqrt[4]{2}} a^3.$$

Similar computations yield analogous conclusions for the more general equation

$$\ddot{x} + x^{2n+1} = 0.$$

3. AN APPLICATION

We now present a type of Cavalieri's principle. Let $f_1 : M_1 \to \mathbb{R}$ and $f_2 : M_2 \to \mathbb{R}$, be maps without critical points and p_1, p_2 the respective "period functions". Suppose that $p_1(z) = p_2(z)$ almost everywhere. Then, given $z_0, z_1 \in \mathbb{R}$, we have Area $(f_1^{-1}(z_0, z_1)) =$ Area $(f_2^{-1}(z_0, z_1))$. In particular, Area $(M_1) =$ Area (M_2).

4. A COAREA FORMULA

As another application, we give a proof of a particular case of the famous coarea theorem (see, [5]).

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THEOREM 4.1. *Let M be an open subset of $\mathbb{R}^2$ and $f : M \to \mathbb{R}$ a C^1 function without critical points. Then* $\int_M J(f(x))\,dx = \int \mu_1(f^{-1}(z))\,dz$, *where $\mu_1(f^{-1}(z))$ is 1-dimensional Hausdorff measure of $f^{-1}(z)$ and $J(f(x))$is the norm of the gradient of f.*

Remark 5: Note that in this case $\mu_1(f^{-1}(z))$ is the the length of $f^{-1}(z)$.
The proof of Theorem 4.1 is a consequence of the lemmas below.

LEMMA 4.1. *Suppose $Jf(x) = c$. Then* $\int_M J(f(x))\,dx = \int \mu_1(f^{-1}(z))\,dz$.

Proof: We have $c\,\text{Area}\,(M) = c\int_M dx = \int_M c\,dx = \int_M Jf(x)\,dx.$
But we also know by Theorem 2, that
$c\,\text{Area}\,(M) = c\int p(z)\,dz = \int c\,p(z)\,dz = \int l(f^{-1}(z))\,dz = \int \mu_1(f^{-1}(z))\,dz,$
and the conclusion follows.

LEMMA 4.2. *Suppose that $Jf(x)$ satisfies $c - \epsilon \le J(f(x)) \le c + \epsilon$, where $0 < \epsilon < c$. Then,*

$$|\int_M J(f(x))\,dx - \int \mu_1(f^{-1}(z))\,dz| \le \frac{2\epsilon}{c-\epsilon}\int \mu_1(f^{-1}(z))\,dz. \tag{7}$$

Proof: We have

$$\int_M (c-\epsilon)\,dx \le \int_M Jf(x)\,dx \le \int_M (c+\epsilon)\,dx.$$

Hence,

$$(c-\epsilon)\ \text{Area}\,(M) \le \int_M Jf(x)\,dx \le (c+\epsilon)\ \text{Area}\,(M).$$

We also have

$$\mu_1(f^{-1}(z)) = \int_0^{p(z)} \|\dot{x}\|\,dt \le (c+\epsilon)p(z) \quad \text{and} \quad \mu_1(f^{-1}(z)) \ge (c-\epsilon)p(z)$$

and these inequalities imply

$$(c-\epsilon)\int p(z)\,dz = \frac{c-\epsilon}{c+\epsilon}\int (c+\epsilon)\,p(z)\,dz \ge \frac{c-\epsilon}{c+\epsilon}\int \mu_1(f^{-1}(z))\,dz.$$

and

$$(c+\epsilon)\int p(z)\,dz = \frac{c+\epsilon}{c-\epsilon}\int (c-\epsilon)\,p(z)\,dz \le \frac{c+\epsilon}{c-\epsilon}\int \mu_1(f^{-1}(z))\,dz.$$

Therefore,

$$\frac{c-\epsilon}{c+\epsilon}\int \mu_1(f^{-1}(z))\,dz \le \int_M J(f(x))\,dx \le \frac{c+\epsilon}{c-\epsilon}\int \mu_1(f^{-1}(z))\,dz.$$

and from these inequalities we get (7).

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LEMMA 4.3. *Let M be a bounded open subset of $\mathbb{R}^2$ and f a C^1 function on $\overline{M}$. Suppose that there is $c > 0$ such that $Jf(x) \geq c$, $\forall x \in M$. Then,*

$$\int_M J(f(x))\,dx = \int \mu_1(f^{-1}(z))\,dz.$$

Proof: Suppose $\epsilon > 0$ is given. Since $J(f(x))$ is continuous, there exist open sets $M_1, M_2, \dots, M_n$ such that, $M - \bigcup_{i=1}^n M_i$ has measure zero and $c_i - \epsilon \leq J(f(x)) \leq c_i + \epsilon$, $\forall x \in M_i$, $i = 1, \dots, n$, where $c_i = J(f(x_i))$ for some $x_i \in M_i$.
Then, denoting $I_i = f(M_i)$, $i = 1, \dots, n$, we have

$$\begin{aligned}\int_M J(f(x))\,dx &= \int_{\cup M_i} J(f(x))\,dx = \sum_{i=1}^n \int_{M_i} J(f(x))\,dx \\ &= \sum_{i=1}^n \int_{I_i} \mu_1(f^{-1}(z))\,dz + \sum_{i=1}^n (\int_{M_i} J(f(x))\,dx - \int_{I_i} \mu_1(f^{-1}(z))\,dz) \\ &= \int_{I_i} \mu_1(f^{-1}(z))\,dz + \sum_{i=1}^n (\int_{M_i} J(f(x))\,dx - \int_{I_i} \mu_1(f^{-1}(z))\,dz).\end{aligned}$$

From Lemma 2 we have:

$$\begin{aligned}\sum_{i=1}^n \Big| \int_{M_i} J(f(x))\,dx - \int_{I_i} \mu_1 f^{-1}(z)\,dz \Big| &\leq \sum_{i=1}^n \frac{2\epsilon}{c_i - \epsilon} \int_{I_i} \mu_1(f^{-1}(z))\,dz \\ &= \frac{2\epsilon}{c - \epsilon} \int \mu_1(f^{-1}(z))\,dz.\end{aligned}$$

Since the integral $\int \mu_1 f^{-1}(z)\,dz$ is finite and $\epsilon > 0$ is arbitrary, we get:

$$\int_M Jf(x)\,dx = \int \mu_1(f^{-1}(z))\,dz.$$

5. AN INEQUALITY FOR THE AREA OF A SURFACE

In the application bellow we get some estimates for the area of the surface using a result due to Lasota and Yorke [8] (see also [1]). The Riemmanian manifold that we consider is a Riemannian sub manifold in a Hilbert space. We recall the following result:

LEMMA 5.1. *Let $x(t)$ be a nontrivial p-periodic solution of the equation $x'(t) = \mu(x(t))$. Assume that μ is Lipschitzian with Lipschitz constant L on the orbit of x. Then $Lp \geq 2\pi$.*

Assume $f\colon M \to \mathbb{R}$ is C^1 and has precisely n critical values $a_1, ..., a_n$ in $(a, b) = f(M)$. For any $z \in I_j = (a_j, a_{j+1})$, $j = 0, 1, ..., n$, where $a_0 = a, a_n = b$, let n_j be the number of corresponding periodic orbits (we recall that for any z, $f^{-1}(z)$ consists of finitely many periodic orbits). Then,

$$L \text{ Area}\,(M) = L \int p(z)\,dz = \sum_{i=1}^n \int_{I_i} Lp(z)\,dz \geq 2\pi \sum_{i=1}^n n_i l(I_i).$$

Therefore, $\text{Area}\,(M) \geq \dfrac{2\pi}{L}\displaystyle\sum_{i=1}^{n} n_i l(I_i)$. Thus we have proved:

THEOREM 5.1. *Suppose $f : M \to \mathbb{R}$ is a C^1 map such that f' is Lipschitzian with Lipschitz constant L. If f has precisely n critical values, then (using the above notation),*

$$\text{Area}\,(M) \geq \frac{2\pi}{L}\sum_{i=1}^{n} n_i l(I_i).$$

In particular, if f has no critical values then,

$$\text{Area}\,(M) \geq \frac{2\pi}{L}(b-a).$$

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