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Polar multiplicities and equisingularity of map germs from $\mathbb{C}^3$ to $\mathbb{C}^3$

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Gaffney showed that if some invariants associated to a family of map germs $f_t : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, are constant along the parameter t then the family is Whitney equisingular.

The number of invariants involved depends on the dimensions (n, p) and this number can be very big according to n and p are big. Then a natural question arises: "For a fixed pair of dimensions (n, p), what is the minimum number of invariants in Gaffney's theorem that are necessary to guarantee the Whitney equisingularity or the topological triviality of family?"

This question was answered in the following cases: $p = 1$ and $n \neq 3$; $n = p = 2$ and $n = 2$, $p = 3$.

In this paper we deal with the case $n = p = 3$. According to Gaffney's result, for the family f_t to be Whitney equisingular we require the constancy of 18 invariants. We reduce this number to 6 for corank 1 germs. We do this by finding relations among invariants and using the fact that these are upper semi-continuous. We apply our result to some families of unimodular germs.
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1. INTRODUCTION

Gaffney studies in [8] the following problem: "Given a 1-parameter family of map germs $\mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, find analytic invariants whose constancy in the family implies the family is Whitney equisingular or topologically trivial."

Gaffney solved this problem in [8] for an important class of finitely determined map germs of discrete stable type. The necessary invariants are the zero stable invariants and the polar multiplicities defined by Teissier in [21]. The results in [8] show that the number of invariants involved depends on the dimensions (n, p) and this number can be very big according to n and p are big. Then a natural question arrises:

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"For a fixed pair of dimensions (n, p), what is the minimum number of invariants in Gaffney's theorem that are necessary to guarante the Whitney equisigularity or the topological triviality of family?"

This question was answered in the following cases. When $p = 1$ and $n \neq 3$, the Milnor number is a complete list of analytic and topologic invariants (see [14] where the approach to the problem in this case is different from that used by Gaffney). In the case $n = p = 2$, the analytic invariants that guarante Whitney equisingularity of the family are the number of cusps, the number of transverse folds and the second polar multiplicity of the discriminant of the germ ([6]). In the case $n = 2, p = 3$, Gaffney showed in [8] that the number of triple points, the number of Whitney umbrellas, the second polar multiplicity of the image of f and the multiplicity of the curve of the double points form a complete list of invariants. Ruas showed in [19] that, in some particular cases, the number of triple points, Whitney umbrellas and the Milnor number of the curve of double points form a complete list of analytic invariants.

In this paper we deal with the case $n = p = 3$. According to Gaffney's result, for the family f_t to be Whitney equisingular we require the constancy of 18 invariants. We reduce this number to 6 for corank 1 germs. We do this by finding relations among the invariants and using the fact that these are upper semi-continuous.

The paper relies heavily on Gaffney's work in [8]. In section 2 we recall some results in [8] and prove our main result in section 3. We apply in this result in section 4 to some unimodular germs given in [5]. We observe here that map-germs $K^3, 0 \to K^3, 0$ ($K = \mathbb{R}$ or $\mathbb{C}$) are studied by several authors ([1], [2], [5], [10], [12], [16]) with applications, for example, to robotics ([4], [11]) and to differential geometry ([18]).

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2. NOTATION AND PRELEMINARIES

We shall denote by $\mathcal{O}(n, p)$ the set of origin preserving germs of holomorphic mappings from $\mathbb{C}^n$ to $\mathbb{C}^p$, $J^k(n, p)$ the set of $k-$jets at the origin of elements of $\mathcal{O}(n, p)$. We let $\mathcal{R}$ denote origin preserving diffeomorphisms of the sourse, $\mathcal{L}$ the corresponding group of the target; $\mathcal{A} = \mathcal{R} \times \mathcal{L}$. There is a natural action of $\mathcal{A}$ on $\mathcal{O}(n, p)$. We let $\mathcal{A}_e$ denote the pseudo group obtained by allowing non origin preseving diffeomorphisms and $\mathcal{O}_e(n, p)$ the set of germs at the origin but not necessarily origin preserving.

We denote the singular set of $f \in \mathcal{O}_e(n, p)$ by $S(f)$. It consists of all points where the rank of the derivative of f is less than $min(n, p)$. We denote the discriminant of f by $\Delta(f)$ and the critical set of f by $\Sigma(f)$. The ideal generated by the set of $p \times p$ minors of the derivative of $f \in \mathcal{O}_e(n, p)$ is denoted by $J(f)$. If we use minors of size k we denote it by $J_k(f)$. If $f \in \mathcal{O}(s + n, p)$ and we only wish to use derivatives with respect to the variables in $\mathbb{C}^n$, we denote the resulting ideal by $J_x(f)$. We denote the determinant of the derivative of $f \in \mathcal{O}_e(n, n)$ by $J[f]$.

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The germs of principal interest in this paper are the finitely $\mathcal{A}$-determined germs. A germs is said to be k-$\mathcal{A}$-determined if any $g \in \mathcal{O}(n,p)$ with the same k-jet as f, i.e. $j^k g = j^k f$, is $\mathcal{A}$-equivalent to f. The germ f is said to be finitely $\mathcal{A}-$determined if it is $k-\mathcal{A}-$determined for some k (see [22], [3] for more on determinacy of map-germs).

Let $f \in \mathcal{O}(n,p)$ and $\overline{f} \in \mathcal{O}(s+n,p)$ such that $\overline{f}|0 \times \mathbb{C}^n = f$, then $\overline{f}$ is called an $s-$parameter deformation of f. An $s-$parameter unfolding of f is a germ $F \in \mathcal{O}(s+n, s+p)$ such that $F(s,x) = (s, \overline{f}(s,x))$ and $\overline{f}(s,x)$ is an $s-$parameter deformation of f. We say that F is a trivial unfolding of f if there are $s-$parameter unfoldings of the identity on $\mathbb{C}^n$ and $\mathbb{C}^p$, say H and K, such that $K \circ F \circ H^{-1} = (id, f)$. If H and K are homeomorphism instead of diffeomorphism, we say that F is topologically trivial. (The above definitions can also be extended to multi-germs.)

Two unfoldings $F, G : \mathbb{C}^s \times \mathbb{C}^n, 0 \to \mathbb{C}^s \times \mathbb{C}^p, 0$ of f are isomorphic if there exist two diffemorphims germs $\phi : \mathbb{C}^s \times \mathbb{C}^n, 0 \to \mathbb{C}^s \times \mathbb{C}^n, 0$ and $\psi : \mathbb{C}^s \times \mathbb{C}^p, 0 \to \mathbb{C}^s \times \mathbb{C}^p, 0$ that are s-parameter unfoldings of the germs of the identities in $\mathbb{C}^n$and $\mathbb{C}^p$ respectively, such that $G = \psi \circ F \circ \phi^{-1}$. Given a germ $h : \mathbb{C}^t, 0 \to \mathbb{C}^s, 0$ we defined the pull-back of F by h, denoted by h^*F, the t-parameter unfolding $h^*F(x,v) = (v, f(x, h(v)))$. The germs F and G are said to be equivalent if there exist a diffeomorphism $h : \mathbb{C}^s, 0 \to \mathbb{C}^s, 0$ such that G is isomorphic to h^*F. We say that t-parameter unfolding G is induced by F if there exist a germ $h : \mathbb{C}^t, 0 \to \mathbb{C}^s, 0$ such that G is isomorphic to h^*F. An unfolding F of f is said to be versal if any other unfolding of f is induced by F. A finitely determined germs admits a versal unfoldings. It admits a trivial versal unfolding if, and only if, it is stable. (See [22].)

Let F be versal unfolding of f. We say that a stable type $\mathcal{Q}$ appears in F if, for any representative $F = (id, f_s(x))$ of F, there exists a point $(s,y) \in \mathbb{C}^s \times \mathbb{C}^p$ such that the germ $f_s : \mathbb{C}^n, S \to \mathbb{C}^p, y$ is a stable germ of type $\mathcal{Q}$ where $S = f^{-1}(y) \cap \Sigma(f_s)$. The call (s,y) and the points (s,x) with $x \in S$ points of stable type $\mathcal{Q}$ in the target and in the source, respectively. If f is stable, we denote the set of points in $\mathbb{C}^s \times \mathbb{C}^p$ of type $\mathcal{Q}$ by $\mathcal{Q}(f)$ and set $\mathcal{Q}_S(f) = f^{-1}(\mathcal{Q}(f)) - \mathcal{Q}_\Sigma(f)$, where $\mathcal{Q}_\Sigma(f)$ denotes $f^{-1}(\mathcal{Q}(f)) \cap \Sigma(f)$.

If f is finitely determined, we denote $\overline{\mathcal{Q}(f)} = (\{0\} \times \mathbb{C}^p) \cap \overline{\mathcal{Q}(F)}$ and $\overline{\mathcal{Q}_S(f)} = (\{0\} \times \mathbb{C}^n) \cap \overline{\mathcal{Q}_S(F)}$, $\overline{\mathcal{Q}_\Sigma(f)} = (\{0\} \times \mathbb{C}^n) \cap \overline{\mathcal{Q}_\Sigma(F)}$.

We say that $\mathcal{Q}$ is a zero-dimensional stable type for the pair (n,p) if $\mathcal{Q}(f)$ has dimension 0 where f is a representative of the stable type $\mathcal{Q}$. We observe that $\mathcal{Q}(F)$ is a close analytic set, since $\overline{\mathcal{Q}(F)} = \cap F(j^{(p+1)} F^{-1}(\overline{\mathcal{A}z_i}))$, where z_i is the $p+1$ jet of the stable type $\mathcal{Q}$ and $\mathcal{A}z_i$ is the $\mathcal{A}-$orbit of z_i.

A finitely determined germ f has discrete stable type if there exist a versal unfolding F of f in which appears only a finite number of stable types. Any finitely determined germ f has a discrete stable type if (n,p) is in the good dimensions, ([17]).

Let $F = (u, \overline{f}(u,x))$ be a 1$-$parameter unfolding of a finitely determined germ f, such that $\overline{f}(u, -)$ preserves the origin for all u. We say that F is a good unfolding of f if there exist neighbourhoods U, W of the origin in $\mathbb{C} \times \mathbb{C}^n$ and $\mathbb{C} \times \mathbb{C}^p$ respectively such that $F^{-1}(W) = U$, F maps $U \cap \Sigma(F) - T$ to $W - T$ and if $(t_0, y_0) \in W - T$ where $S = F^{-1}(t_0, y_0) \cap \Sigma(F)$ and $T = \mathbb{C} \times \{0\}$, then the germ $f_{t_0} : \mathbb{C}^n, S \to \mathbb{C}^p, y_0$ is stable. It is shown in [8] that the stable types in the source and in the target form a regular stratification, i.e. satisfy the Whitney conditions, therefore the family F is Whitney equisingular.

Suppose that $\mathcal{Q}(F) = \{p_1, \cdots, p_r\}$ is the set of points of zero-dimensional type, where F is a versal unfolding of f. The $0-$stable invariant of type $\mathcal{Q}$ of f, denoted by $m(f;\mathcal{Q})$ is the multiplicity of the ideal $m_s\mathcal{O}_{\overline{\mathcal{Q}(F)},(0,0)}$ in $\mathcal{O}_{\overline{\mathcal{Q}(F)},(0,0)}$.

A good unfolding is said to be excellent if all the $0-$stable invariants are constant in the unfolding and f is of discrete type. In the equidimensional case $(n = p)$, it is also assumed that the degree of f, $\delta(f) = dim_{\mathbb{C}} \frac{\mathcal{O}_n}{f^*(m_n)\mathcal{O}_n}$, is constant in the unfolding.

Using the polar multiplicities defined by Teissier in [21] of the polar varieties of the stable types and Thom's isotopy lemmas, Gaffney showed the following principal result.

Theorem 2.2.1. ([8]) *Suppose that $F : \mathbb{C} \times \mathbb{C}^n, (0,0) \to \mathbb{C} \times \mathbb{C}^p, (0,0)$ is an excellent unfolding of a finitely determined germ $f \in \mathcal{O}(n,p)$. Also suppose that the polar invariants of all the stable types defined in $\Delta(f)$, $\Sigma(f)$ and $f^{-1}(\Delta(f)) - \Sigma(f)$ are constant at the origin for f_t. Then the unfolding is Whitney equisingular.*

In the next section we shall need the following results.

Theorem 2.2.2. (Lê-Greuel's theorem, [13]) *Let X_1 be an I.C.I.S. with a singularity at $0 \in \mathbb{C}^n$. Let X be an I.C.I.S. defined in X_1 by $f_k = 0$, and let $f_1, ..., f_{k-1}$ be the gerators of the ideal that defines X_1 at 0 in $\mathbb{C}^n$. Then*

$$\mu(X_1, 0) + \mu(X, 0) = dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(f_1, ..., f_{k-1}, J(f_1, ..., f_k))}$$

Theorem 2.2.3. (Looijenga's theorem, [15]) *Let $f : \mathbb{C}^k, 0 \to \mathbb{C}^k, 0$ is a germ such that $X = f^{-1}(0)$ is an I.C.I.S. Then $\mu(X, 0) = \delta(f) - 1$.*

Theorem 2.2.4. ([7]). *Let $f : \mathbb{C}^n, 0 \to \mathbb{C}^n, 0$ be a finitely determined germ. Then $f : \Sigma(f) \subset \mathbb{C}^n, 0 \to \Delta(f) \subset \mathbb{C}^n, 0$ is bimereomorphic.*

3. EQUISINGULARITY OF MAP GERMS IN $\mathcal{O}(3,3)$

3.1. The stable types in $\mathcal{O}(3,3)$

As heiglighted in the introduction, our aim is to minimize the number of invariants defined in the stable types of f whose constancy in the family f_t implies the family is topologically trivial or Whitney equisingular.

The strategy is to apply Theorem 2.2.1 and the techniques used by Gaffney in [8], that is, stratify the source and the target by the stable types and establish relations among the invariants on the strata. As these invariants are upper semi-continuous, the relations will allow us to reduce the number of invariants required in Gaffney's theorem.

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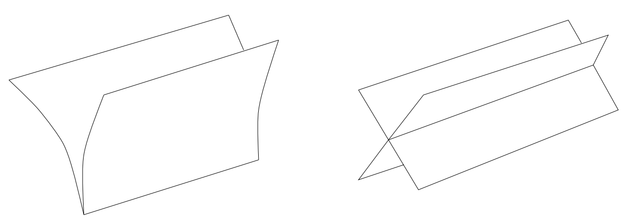
Let $f \in \mathcal{O}(3,3)$ be a finitely determined germ. As $(n,p) = (3,3)$ belongs to the nice dimensions, then f is of discrete type. The stratification in the source and target are as follows.

In the source: We have the set of critical points $\Sigma(f)$, the cuspidal edge curve, denoted by $\Sigma^{1,1}(f)$, and the set of double points $D^2(f|\Sigma(f))$. Due to the complicated structure of $D^2(f|\Sigma(f))$ in general, we shall restrict to corank 1 germs. In this case we consider instead the $D^2_1(f|\Sigma(f))$ which is defined as follows. Let $f|\Sigma(f)$ be the restriction of f to $\Sigma(f)$, then the set of double points of f, denoted by $D^2(f|\Sigma(f))$, is the set of points $(p_1, p_2) \in \mathbb{C}^6$, $p_1 \neq p_2$, such that $p_1, p_2 \in \Sigma(f)$ and $f(p_1) = f(p_2)$. If f is of corank 1, we can write $f = (x, y, g(x, y, z))$, and consider $D^2(f|\Sigma(f))$ (see [10] for details) as the subset

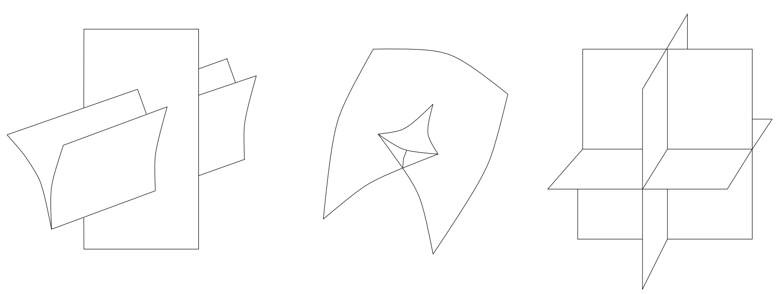
$$\{(x,y,z,z_1) \in \mathbb{C}^4 : \frac{g(x,y,z)-g(x,y,z_1)}{z-z_1} = \frac{z_1\frac{\partial g(x,y,z)}{\partial z} - z\frac{\partial g(x,y,z_1)}{\partial z}}{z-z_1} = \frac{\frac{\partial g(x,y,z)}{\partial z} - \frac{\partial g(x,y,z_1)}{\partial z}}{z-z_1} = 0\}.$$

Then we denoted by $D^2_1(f|\Sigma(f))$ the projection of $D^2(f|\Sigma(f))$ to the (x, y, z_1)-space.

In the target: We have the discriminant of f, $\Delta(f) = f(\Sigma(f))$, the image of the cuspidal edge curve $f(\Sigma^{1,1}(f))$, the image of the double points curve $f(D^2_1(f|\Sigma(f)))$ (for corank 1 germs), and the zero-dimensional stable types. These are the swallowtail points (A_3), normal crossing of a plane with a cuspidal edge (A_1A_2) and the set of triple points (A_1^3) (see Figure 1).



Stable types of dimension 1



Stable types of dimension 0

FIG. 1. The stable types in the target

To a $k-$dimensional variety are associated $k+1$ polar invariants. As $\Sigma(f)$ and $\Delta(f)$ are of dimension 2 and the dimension of $D^2_1(f|\Sigma(f))$, $\Sigma^{1,1}(f)$, $f(\Sigma^{1,1}(f))$ and $f(D^2_1(f|\Sigma(f)))$ is 1, we have 14 polar invariants defined on these sets. We also have 3 multiplicities of the zero-dimensional stable types. These are the number swallowtails ($\sharp A_3$), the number of normal crossing of a plane with a cuspidal edge ($\sharp A_1A_2$) and the number of triple points

$(\sharp A_1^3)$. Therefore to apply Theorem 2.2.1 to germs in $\mathcal{O}(3,3)$ we needed the constancy of 18 invariants, including the degree of f. In the following sections we show that some of these invariants are related.

3.2. Relations among the invariants of the stable types in the target

We start by analysing the discriminant $\Delta(f)$. We have the following general result.

THEOREM 3.3.1. *Let $f \in \mathcal{O}(n,n)$ be a finitely determined germ and $p : \mathbb{C}^n \to \mathbb{C}^{n-1}$ a generic linear projection. Then*

$$m_0(\Delta(f)) = \mu(p \circ f) + \delta(f) - 1.$$

Proof: We choose a projection $p : \mathbb{C}^n, 0 \to \mathbb{C}^{n-1}, 0$ such that the degree of $p|\Delta(f)$ is the multiplicity of $\Delta(f)$ at the origin. Let

$$X_1 = (p \circ f)^{-1}(0),$$

$$X = f^{-1}(0).$$

As these spaces are isolated complete intersections singularities (I.C.I.S.), by appling Lê-Greuel's theorem we have

$$\mu(p \circ f) + \mu(f_1, ..., f_n) = dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(p \circ f, J[f])}.$$

Also, by Looijenga's theorem we have

$$\mu(f_1, ..., f_n) = \delta(f) - 1.$$

The idea now is to use $\Sigma(f)$ rather than $\Delta(f)$ as the later has non-isolated singularities and the former is a hypersurface with isolated singularity. Since $f : \Sigma(f) \subset \mathbb{C}^n \to \Delta(f) \subset \mathbb{C}^n$ is bimeoromorphic and finite ([7]), the local ring $\mathcal{O}_{\Sigma(f)}$ is Cohen-Macaulay and the diagram below comutates. Therefore the degree of $p \circ f|\Sigma(f)$ is equal to the degree of $p|\Delta(f)$ which is the polar multiplicity of $\Delta(f)$. Hence, $dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(p \circ f, J[f])} = m_0(\Delta(f))$, and the result follows

$$\begin{array}{ccc} \Sigma(f) \subset \mathbb{C}^n & \xrightarrow{f} & \Delta(f) \subset \mathbb{C}^n \\ p \circ f \searrow & & \swarrow p \\ & (\mathbb{C}^{n-1}, 0) & \end{array}$$

$\square$

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Corollary 3.3.2. *Let $f \in \mathcal{O}(3,3)$ be a finitely determined germ and $p : \mathbb{C}^3 \to \mathbb{C}^2$ a generic linear projection. Then*

$$m_0(\Delta(f)) = \mu(p \circ f) + \delta(f) - 1.$$

Remark 3.3.3. *The constancy of both $m_0(\Delta(f_t))$ and $\delta(f_t)$ is required when applying Gaffney's theorem. Using the relation in the above Corollary, we only need the constancy of one of them.*

Corollary 3.3.4. *If $f \in \mathcal{O}(3,3)$ if of corank 1, then $m_0(\Delta(f)) = \delta(f) - 1$.*

Theorem 3.3.5. *Let $f \in \mathcal{O}(3,3)$ be a finitely determined germ. Then*

$$m_2(\Delta(f)) - m_1(\Delta(f)) + m_0(\Delta(f)) = \mu(\Sigma(f)) + 1.$$

Proof: We choose a projection $p_2 : \mathbb{C}^3 \to \mathbb{C}^2$ such that the degree of $p_2|\Delta(f)$ is equal to the multiplicity of $\Delta(f)$ at the origin, and the multiplicity of the polar variety $P_1(\Delta(f)) = \overline{\Sigma(p_2|\Delta(f)^0)}$ is $m_1(\Delta(f))$. Let $p_1 : \mathbb{C}^2 \to \mathbb{C}$ be a generic linear projection such that the degree of $p_1 \circ p_2|P_1(\Delta(f))$ is $m_1(\Delta(f))$. We can suppose that the projection $p_1 \circ p_2$ is also generic and defines the polar multiplicity $m_2(\Delta(f))$. Let

$$X_1 = V(p_1 \circ p_2 \circ f, J[f]),$$

$$X = V(p_2 \circ f, J[f]).$$

As these varieties are I.C.I.S., we apply Lê-Greuel's theorem and obtain

$$\mu(X_1) + \mu(X) = dim_{\mathbb{C}} \ \frac{\mathcal{O}_3}{(p_1 \circ p_2 \circ f, J[f], J[p_2 \circ f, J[f]])}.$$

Applying Lê-Greuel's again to $Y = V(J[f])$ and X_1, we have

$$dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(J[f], J(p_1 \circ p_2 \circ f, J[f]))} - \mu(J[f]) + \mu(X) = dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(p_1 \circ p_2 \circ f, J[f], J[p_2 \circ f, J[f]])}.$$

By Looijenga's theorem, we have $\mu(X) = deg(p_2 o f, J[f]) - 1$. Therefore

$$dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(J[f], J(p_1 \circ p_2 o f, J[f]))} - \mu(J[f]) + deg(p_2 \circ f, J[f]) - 1 = dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(p_1 \circ p_2 \circ f, J[f], J[p_2 \circ f, J[f]])}. \qquad (1)$$

This equation is the key to establish the relation among the polar multiplicities of the discriminant. To calculate the multiplicity $m_0(\Delta(f))$ we use the following comutative diagram

$$\begin{array}{ccc} \Sigma(f) \subset \mathbb{C}^3 & \xrightarrow{f} & \Delta(f) \subset \mathbb{C}^3 \\ p_2 \circ f \searrow & & \swarrow p_2 \\ & (\mathbb{C}^2, 0) & \end{array}$$

Since $f : \Sigma(f) \to \Delta(f)$ is finite and bimeoromorphic $deg(p_2|\Delta(f)) = deg(p_2 \circ f|\Sigma(f))$ and therefore $deg(p_2 \circ f, J[f]) = m_0(\Delta(f))$. In the same way, to calculate $m_1(\Delta(f))$, we used the fact that f restricted to $V' = V(J[f], J[p_2 \circ f, J[f]])$ is finite and bimeoromorphic. Then we have $deg((p_1 \circ p_2 \circ f)|V') = deg((p_1 \circ p_2)|\Delta(f))$ and

$$m_1(\Delta(f)) = dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(p_1 \circ p_2 \circ f, J[f], J[p_2 \circ f, J[f]])}.$$

This last equality follows from the fact that the ring $\mathcal{O}_{V'}$ is Cohen-Macaulay. To calculate $m_2(\Delta(f))$, we choose a versal unfolding F of f and consider $F : \Sigma(f) \subset \mathbb{C}^3 \times \mathbb{C}^s \to \Delta(f) \subset \mathbb{C}^3 \times \mathbb{C}^s$. We know that $\Sigma(F)$ is a hypersurface in $\mathbb{C}^3 \times \mathbb{C}^s$. As $p_1 \circ p_2 : \mathbb{C}^3 \to \mathbb{C}$ is a generic linear projection we have $\Sigma((\pi_s, p_1 \circ p_2) \circ F)|\Sigma(F)) = V(J[F], J[p_1 \circ p_2 \circ F, J[F]])$.

The invariant $m_2(\Delta(f))$ is controled by the degree of the projection $\pi_s|V(J[F], J[p_1 \circ p_2 \circ F, J[F]])$, that is, by the colength $e_J(f)$ of the maximal ideal m_s in $\mathcal{O}_3$ of the source. This is given by

$$e_J(f) = dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(J[f], J(p_1 \circ p_2 \circ f, J[f]))}.$$

The possible components of $V(J[F], J[p_1 \circ p_2 \circ F, J[F]])$ are the closures of the following sets $F^{-1}(P_2(\Delta(F), \pi_s))$, $F^{-1}(A_3)$, $F^{-1}(A_1A_2)$ and $F^{-1}(A_1^3)$.

To count the contribution of π_s restricted to these components, we use the normal forms of the stable types. We choose a generic parameter s and neighbourhoods $U_2 \subset \mathbb{C}^3 \times \mathbb{C}^s$and $U_1 \subset \mathbb{C}^s$ such that for every point in U_1 we have $e_J(f)$ pre-images in $V \cap U_2$ counting multiplicity. Then

$$\begin{aligned} e_J(f) &= \textstyle\sum_{x \in S} dim_{\mathbb{C}} \frac{\mathcal{O}_{s+3,x}}{(m_s, J[F], J(p_1 \circ p_2 \circ F, J[F]))} \\ &= \textstyle\sum_{x \in S} dim_{\mathbb{C}} \frac{\mathcal{O}_{3,x}}{(J[f_s], J(p_1 \circ p_2 \circ f_s, J[f_s]))}. \end{aligned}$$

where $S = {\pi_s}^{-1}(0) \cap V$. As the parameter s is generic, we can suppose that f_s is stable. First we choose a germ where one singularity of type A_3 appears. This has normal form $f_s = (x, y, z^4 + xz + yz^2)$ and we count its contribution in the variety V. We have

$$dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(4z^3+x+2yz, J(y, 4z^3+x+2yz))} = dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(4z^3+x+2yz, 1, 12z+2y)} = 0$$

Therefore the contribution of the points A_3 in $e_J(f)$ is nil, this means that these points do not appear in V.

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In the same way we consider the normal form $f_s = \{(x, y, z^3 + yz); (x^2, y, z)\}$ for the A_1A_2 singularity and the normal $f_s = \{(x, y, z^2);\ (x, y^2, z);\ (x^2, y, z)\}$ for the singularity A_1^3, and show that their contributions are also nil. Thus, the components of $V(J[F], J[p_1 \circ p_2 \circ F, J[F]])$ are in the closure of $F^{-1}(P_2(\Delta(F), \pi_s))$, so

$$\begin{aligned} e_J(f) &= \textstyle\sum_{x \in S} dim_{\mathbb{C}}\ \frac{\mathcal{O}_{3,x}}{(J[f_s], J(p_1 \circ p_2 \circ f_s, J[f_s]))} \\ &= m_2(\Delta(f)). \end{aligned}$$

As F restricted to $V(J[F], J[p_1 \circ p_2 \circ F, J[F]])$ is finite and bimeoromorphic, the result follows by substituiting all the above equalities in equation (1). □

Remark 3.3.6. *As the polar multiplicities are upper semi-continuous, the relation in Theorem 3.3.5 shows that if $m_0(\Delta(f_t))$ and $\mu(\Sigma(f_t))$ are constant in the family then $m_1(\Delta(f_t))$ and $m_2(\Delta(f_t))$ are also constant in the family.*

Corollary 3.3.7. *Let $f \in \mathcal{O}(3, 3)$ be a finitely determined germ of corank 1. Then*

$$\begin{gathered} m_2(\Delta(f)) - m_1(\Delta(f)) + m_0(\Delta(f)) \\ = \\ \sharp A_1A_2 + 2\sharp A_1^3 - \tfrac{1}{2}\sharp A_3 + \mu_\Delta(f) - \tfrac{1}{2}(\mu(D_1^2(f|\Sigma(f)))) + \tfrac{3}{2}. \end{gathered}$$

Proof: The proof follows by applying Theorem 3.3.5 and to the following results of Huston in [12]. If $f \in \mathcal{O}(3, 3)$ is of corank 1, then

$$\mu_\Delta(f) = \mu(\Sigma(f)) + \frac{1}{2}(\mu(D^2(f|\Sigma(f))) + \sharp A_3 - 1) + \sharp A_1^3,$$

$$\mu(D_1^2(f|\Sigma(f))) = \mu(D^2(f|\Sigma(f)) + 2\sharp A_1A_2 + 6\sharp A_1^3$$

□

We shall now consider the set $f(D_1^2(f|\Sigma(f)))$. The situation is more complicated when dealing with the double point set as its structure is unknown in general. However, for finitely determined germs of corank 1, Goryunov showed in [10] that $D_1^2(f|\Sigma(f))$ is an I.C.I.S.

Theorem 3.3.8. *Let $f \in \mathcal{O}(3, 3)$ be a finitely determined germ of corank 1. Then*

$$2m_0(X) - 2m_1(X) + \mu(D_1^2(f|\Sigma(f))) = 2\sharp A_1A_2 + 3\sharp A_3 + 1,$$

where $X = f(D_1^2(f|\Sigma(f)))$.

Proof: The stratum $f(D_1^2(f|\Sigma(f)))$ has dimension 1 so we need to consider two polar invariants $m_0(f(D_1^2(f|\Sigma(f))))$ and $m_1(f(D_1^2(f|\Sigma(f))))$.

As $f(D_1^2(f|\Sigma(f)) - \{0\})$ is a two fold cover of $f(D_1^2(f)) - \{0\}$ ([23] page 294), we can choose a projection in such a way that

$$e(f^*(m_3)\mathcal{O}_{D_1^2(f)}) = 2m_0(f(D_1^2(f|\Sigma(f))),$$

where $e(I)$ denoted the multiplicity of the ideal I. The set $D_1^2(f|\Sigma(f))$ is an I.C.I.S., so the following sets are also I.C.I.S.

$$X_1 = D_1^2(f|\Sigma(f)),$$

$$X_2 = D_1^2(f|\Sigma(f)) \cap (p \circ f)^{-1}(0)$$

where $p : \mathbb{C}^3 \to \mathbb{C}$ is a generic linear projection. Applying Lê-Greuel's theorem to these sets we obtain

$$\mu(X_1) + \mu(X_2) = e(I_1^2(f|\Sigma(f)), J[I_1^2(f|\Sigma(f)), p \circ f]) \tag{2}$$

where $I_1^2(f|\Sigma(f))$ is the ideal that defines the curve of double points $D_1^2(f|\Sigma(f))$. As X_2 is I.C.I.S. of dimension zero, we can apply Looijenga's theorem to obtain

$$\mu(X_2) = e(I_1^2(f|\Sigma(f)), p \circ f) - 1.$$

From the previous equalities and the fact that f restricted the double point curve is bimeoromorphic, we have

$$\mu(X_2) = 2m_0(f(D_1^2(f|\Sigma(f)))) - 1.$$

To understand the geometric meaning of the right hand side of the equality (2) we choose a versal unfolding $F = (s, \overline{f}(s,x))$ of f and consider the variety $V(I_1^2(F|\Sigma(F)), J[I_1^2(F|\Sigma(F)), p \circ \overline{f}])$. The components of this variety is the closure of the sets $F^{-1}(P_1(D_1^2(F|\Sigma(F))))$, the set $F^{-1}(A_1A_2)$ points, the set of $F^{-1}(A_3)$ points and the set of triples points A_1^3. That have dimension at least s.

For a generic projection p, the dimension of V is s. As the multiplicity of V is the sum of the multiplicities of these components, it is enough to calculate the contribution of the degree of π_s restricted to each component, where π_s is the projection of the first component of $\mathbb{C}^s \times \mathbb{C}^3$ to $\mathbb{C}^s$.

We choose neighbourhoods U_1 of 0 in $\mathbb{C}^s$ and U_2 of 0 in $\mathbb{C}^s \times \mathbb{C}^3$ such that on each point in U_1, π_s has $e(I_1^2(f|\Sigma(f)), J[I_1^2(f|\Sigma(f)), p \circ f])$ pre-images in $V \cap U_2$ counting multiplicity. If $s \in \mathbb{C}^s$ is a generic parameter close the origin we have

$$\begin{aligned} e(H) &= \textstyle\sum_{x \in S} dim_{\mathbb{C}} \frac{\mathcal{O}_{s+3,x}}{(m_s, I_1^2(F|\Sigma(F)), J[I_1^2(F|\Sigma(F)), p\circ\overline{f}])} \\ &= \textstyle\sum_{x \in S} dim_{\mathbb{C}} \frac{\mathcal{O}_{3,x}}{(I_1^2(f_s|\Sigma(f_s)), J[I_1^2(f_s|\Sigma(f_s)), p\circ f_s])} \end{aligned} \tag{3}$$

where $H = (I_1^2(f|\Sigma(f)), J[I_1^2(f|\Sigma(f)), p \circ f])$ and $S = \pi_s^{-1}(0) \cap V$. To count the contribution of singularities of type A_3, we take $f_s = (x, y, z^4 + xz + yz^2)$.

As $\Sigma(f)$ is a regular hypersurface, $f|\Sigma(f)$ can be considered as a map from $\mathbb{C}^2 \to \mathbb{C}^3$. The discriminant is parametrised by $\phi(y,z) = (-4z^3 - 2yz, y, -3z^3 - yz^2)$. The double points curve of this application, denoted by $D^2(\phi) \subset \mathbb{C}^3$, is give by the ideal

$$I = (-4(z_1^2 + z_1 z + z^2) - 2y, -(z_1^2 + z^2)(z_1 + z) - y(z_1 + z)).$$

This curve is bimeoromophic to the curve $D_1^2(f|\Sigma(f))$. Then, to calculate the contribution of the singularities of type A_3 in $V(D_1^2(f|\Sigma(f)), J[D_1^2(f|\Sigma(f)), p \circ f])$ it is enough to use the ideal I. Using the formula (3) we have

$$e(H) = \sum_{x \in S} dim_{\mathbb{C}} \frac{\mathcal{O}_{3,x}}{(I, J[I, ax+by+cf_3])} = 3,$$

where $p(x,y,z) = ax + by + cz$ is a generic projection ($b \neq 0$). This means that the points of type A_3 appear in V with a contribution equals to 3.

In the same way, to count the contribution of the points A_1A_2 it is enough to choose the normal form $f_s = \{(x^2, y, z); (x, y, z^3 + yz)\}$. In fact, the double points curve in the source of this multigerm is defined by the ideal $I = (x, 3z^2 + y)$. Then using (3), we have

$$\begin{aligned} e(H) &= \sum_{x \in S_1} dim_{\mathbb{C}} \frac{\mathcal{O}_{3,x}}{(x, 3z^2+y, J[x, 3z^2+y, ax+by+cf_3])} \\ &\quad + \sum_{x \in S_2} dim_{\mathbb{C}} \frac{\mathcal{O}_{3,x}}{(x, 3z^2+y, J[x, 3z^2+y, ax+by+cf_3])} \\ &= 2, \end{aligned}$$

where $S_1 = \pi_s^{-1}(0) \cap V_1$ and $S_2 = \pi_s^{-1}(0) \cap V_2$ for each component of the multigerm and $b \neq 0$. This means that the contribution of the points A_1A_2 is 2.

Since F restricted to each component is finite and bimeoromorphic, we have

$$deg(\pi_s|V) = 2m_1(f(D_1^2(f))) + 3\sharp A_3 + 2\sharp A_1A_2.$$

The theorem follows now by joining all the above equalities. □

We now consider the set $f(\Sigma^{1,1}(f))$. Just as in the previous theorem we needed the set $f(\Sigma^{1,1}(f))$ to be an I.C.I.S., so we considered f to be of corank 1.

THEOREM 3.3.9. *Let $f \in \mathcal{O}(3,3)$ be a finitely determined germ of corank* 1. *Then*

$$m_0(f(\Sigma^{1,1}(f))) - m_1(f(\Sigma^{1,1}(f))) = \sharp A_3 - \mu(\Sigma^{1,1}(f))) + 1$$

Proof: Let F be a versal unfolding of f. Then $\Sigma^{1,1}(F)$ has a reduced structure. As $\Sigma^{1,1}(f)$ is an analytic space that is I.C.I.S., we choose $p : \mathbb{C}^3 \to \mathbb{C}$ a generic linear projection such that the spaces

$$X_1 = \Sigma^{1,1}(f),$$

$$X = \Sigma^{1,1}(f) \cap (p \circ f)^{-1}(0)$$

are I.C.I.S.. Applying Lê-Greuel's theorem, we obtain

$$\mu(\Sigma^{1,1}(f)) + \mu(\Sigma^{1,1}(f) \cap (p \circ f)^{-1}) = e(I^{1,1}(f), J[I^{1,1}(f), p \circ f])$$

where $I^{1,1}(f)$ is the ideal that defines $\Sigma^{1,1}(f)$.

As X is of dimension zero, we can apply Looijenga's theorem and as $f|\Sigma^{1,1}(f)$ is bimeromorphic, to obtain

$$\mu(I^{1,1}(f), p \circ f) = m_0(f(\Sigma^{1,1}(f))) - 1.$$

We need to understand the geometric meaning of

$$e(I^{1,1}(f), J[I^{1,1}(f), p \circ f]).$$

For this, we choose a versal unfolding $F = (s, \overline{f}(x,s))$ of f and considered the variety $V(I^{1,1}(F), J[I^{1,1}(F), p \circ \overline{f}])$. The components of this variety is the closure of the sets : $F^{-1}(P_1(f(\Sigma^{1,1}(F))))$ and the pre-images of the swallowtail singularity A_3. From the genericity of p and F, these components have dimension exactly s. As F restricted to each component is finite and bimeoromorphic, we have

$$deg(\pi_s|V) = m_1(f(\Sigma^{1,1}(f))) + \sharp A_3.$$

The contributions are obtained in the same way as in the previous theorem. □

Corollary 3.3.10. *Let $f \in \mathcal{O}(3,3)$ be a finitely determined germ of corank 1. Then*

$$m_1(\Delta(f)) = m_0(f(\Sigma^{1,1}(f))).$$

Proof: As f is of corank 1 we can write, $f = (x, y, g(x,y,z))$ and the ideal that defines $\Sigma^{1,1}(f)$ is $I^{1,1}(f) = (g_z, g_{zz})$. In the proof of Theorem 3.3.5 and Theorem 3.3.9, we have

$$m_1(\Delta(f)) = dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(p_1 \circ p_2 \circ f, J[f], J[p_2 \circ f, J[f]])}$$

and

$$m_0(f(\Sigma^{1,1}(f))) = dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(I^{1,1}(f), p \circ f)}.$$

As p_1, p_2 and p are generic projections,

$$dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(p_1 \circ p_2 \circ f, J[f], J[p_2 \circ f, J[f]])} = dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(I^{1,1}(f), p \circ f)}.$$

□

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3.3. Relations among the invariants of the stable types in the source

In this section we stablish relations among the invariants of the stable types in the source. The strata in this case are the set of critical points, the curve of double points and the cuspidal edge curve. The situation is less difficult than in the case of the target because the set of critical points is a hypersurface with isolated sigularity, and for a germ of corank 1 the double points curve and the cuspidal edge curve are I.C.I.S.

We know from [20] that the absolute polar multiplicities of a hypersurface X with isolated singularity are related to the Milnor numbers μ^k, of the plane sections by the following equalities

$$m_k(X) = \mu^{k+1}(X) + \mu^k(X)$$

for $0 \leq k \leq d-1$, where $d = dim(X)$. This result is also valid for I.C.I.S. (see [13],[9]). The absolute polar multiplicities are defined when the dimension of X is ≥ 1. The multiplicity $m_d(X)$ cannot be defined directly like the other m_k, $0 \leq k \leq d-1$, because the singularities of $p_1|X$ are isolated points. However, Gaffney [9] defines this multiplicity for spaces that are I.C.I.S as follows.

Definition 3.3.11. *The $d-th$ polar multiplicity of $(X^d, 0)$ (X^d is I.C.I.S. of dimension d), denoted by $m_d(X^d)$, is defined by*

$$m_d(X^d) = dim_{\mathbb{C}} \frac{\mathcal{O}_X}{J(p_1, f)}$$

where $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n-d}, 0)$, $f^{-1}(0) = X^d$ and $p_1 : \mathbb{C}^n \to \mathbb{C}$ is a generic linear projection.

Remark 3.3.12. *As $V(p_1, f)$ is I.C.I.S., then by Lê-Greuel's theorem, we have*

$$m_d(X^d) = \mu(X^d) + \mu(X^d \cap p_1^{-1}(0)).$$

When $f \in \mathcal{O}(3,3)$ is finitely determined, $\Sigma(f)$ is a hypersurface with an isolated singularity. If f is of corank 1, the double points curve $D_1^2(f|\Sigma(f))$ and the cuspidal edge curve $\Sigma^{1,1}(f)$ are I.C.I.S. Therefore we can apply Definition 3.3.11 and all the properties above to obtain the following.

Proposition 3.3.13. *Let $f \in \mathcal{O}(3,3)$ be a finitely determined germ. Then*

(i)

$$m_2(\Sigma(f)) - m_1(\Sigma(f)) + m_0(\Sigma(f)) = \mu(\Sigma(f)) + 1$$

(ii)If f is of corank 1, then

$$m_1(D_1^2(f|\Sigma(f))) - m_0(D_1^2(f|\Sigma(f))) = \mu(D_1^2(f|\Sigma(f))) - 1$$
$$m_1(\Sigma^{1,1}(f)) - m_0(\Sigma^{1,1}(f)) = \mu(\Sigma^{1,1}(f))) - 1$$

We can now deduce our main theorem. Using the results of subsections 3.2 and 3.3, we reduce the number of invariants in Gaffney's theorem 2.2.1 from 18 to 6 in the corank 1 case.

THEOREM 3.3.14. *Suppose that* $f \in \mathcal{O}(3,3)$ *is a finitely determined germ of corank 1 and* $F = (t, f_t)$ *is a good* $1-$*parameter unfolding. Then* F *is Whitney equisingular along* $T = \mathbb{C} \times \{0\}$ *if, and only if,* $m_1(\Delta(f_t))$, $\mu(\Sigma(f_t))$, $m_1(\Sigma(f_t))$, $m_1(D_1^2(f_t|\Sigma(f_t)))$, $m_0(f_t(D_1^2(f_t|\Sigma(f_t))))$ *and* $m_1(\Sigma^{1,1}(f_t))$ *are constant for* t *close to the origin.*

THEOREM 3.3.15. *Suppose that* $f \in \mathcal{O}(3,3)$ *is a finitely determined germ and* F *an excellent unfolding of* f*. Then* F *is Whitney equisingular along* $T = \mathbb{C} \times \{0\}$ *if, and only if,* $m_1(\Delta(f_t))$, $\mu(\Sigma(f_t))$, $m_1(\Sigma(f_t))$ *and the polar multiplicities of the* $1-$*dimensional stable types in the source and in the target are constant for* t *close to the origin.*

4. APPLICATION

In an unpublished work, du Plessis and Tari [5] classified unimodular germs from $\mathbb{C}^3 \to \mathbb{C}^3$ that are inside the $\mathcal{K}-$orbit A_3. The models for these germs are the following:

$$\begin{array}{ll} (x, y, z^4 + (xy + ay^3 + y^4)z + xz^2) & a \neq 0, 1/2 \\ (x, y, z^4 + (xy + ay^3)z + xz^2) & a \neq 0, 1/2 \\ (x, y, z^4 + (x^2 + xy^2 + ay^4)z + xz^2) & a \neq 0, 1/4 \end{array}$$

We apply in this section the main result (Theorem 3.3.14) to the study of the equisingularity of the above germs along the parameter modulus a.

PROPOSITION 4.4.1. *Let* $F(t, x, y, z) = (t, x, y, z^4 + (xy + (a + t)y^3 + y^4)z + xz^2)$ *and* $G(t, x, y, z) = (t, x, y, z^4 + (xy + (a + t)y^3)z + xz^2)$ *be unfoldings of the germs,* $f(x, y, z) = (x, y, z^4 + (xy + ay^3 + y^4)z + xz^2)$ *and* $g(x, y, z) = (x, y, z^4 + (xy + ay^3)z + xz^2)$ *respectively. Then* F *and* G *are Whitney equisingular for* t *sufficiently small if* $a \neq 0, 1/2$.

Proof: Let $f_t(x, y, z) = (x, y, z^4 + (xy + (a + t)y^3 + y^4)z + xz^2)$ be a deformation of f. According to Theorem 3.3.14 we have to calculate 6 invariants. We calculate first the polar multiplicity of the discriminant $m_1(\Delta(f_t))$:

$$m_1(\Delta(f_t)) = dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(p_1 \circ p_2 \circ f_t, J[f_t], J[p_2 \circ f_t, J[f_t]])}$$

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where p_1 and p_2 are generic projections. We take $p_1(x, y) = y$ and $p_2(x, x, z) = (x, y)$ which are generic. Then

$$\begin{aligned} m_1(\Delta(f_t)) &= dim_{\mathbb{C}} \tfrac{\mathcal{O}_3}{(y, 4z^3+xy+(a+t)y^3+y^4+2xz, J[x,y,4z^3+xy+(a+t)y^3+y^4+2xz])} \\ &= dim_{\mathbb{C}} \tfrac{\mathcal{O}_3}{(y,4z^3+xy+(a+t)y^3+y^4+2xz, 12z^2+2x)} \\ &= dim_{\mathbb{C}} \tfrac{\mathcal{O}_2}{(4z^3+2xz, 12z^2+2x)} \\ &= 3 \\ &= m_1(\Delta(f_0)) \end{aligned}$$

We have $\Sigma(f_t) = 4z^3 + xy + (a + t)y^3 + y^4 + 2xz$. As this is a hypersurface with isolated singularity its Milnor number is well defined and is given by $\mu(\Sigma(f_t)) = 2$, if $(a+t) \neq 1/2$. Then for $a \neq 1/2$, and for t small $\mu(\Sigma(f_t)) = \mu(\Sigma(f_0))$.

We compute now the multiplicity $m_1(\Sigma(f_t)))$. As $\Sigma(f_t)$ is a hypersurface with isolated singularity,

$$m_1(\Sigma(f_t)) = \mu^2(\Sigma(f_t)) + \mu^1(\Sigma(f_t)).$$

We choose generic hyperplanes $H_1 = \{x = y = 0\}$ and $H_2 = \{y = 0\}$ so that,

$$m_1(\Sigma(f_t)) = \mu^2(4z^3 + 2xz) + \mu^1(4z^3) = 3 = m_1(\Sigma(f_0)).$$

To calculate the multiplicity $m_1(D_1^2(f_t|\Sigma(f_t)))$ we needed to find the ideal that defines the curve $D_1^2(f_t|\Sigma(f_t))$ and use the following relation from Proposition 3.3.13

$$m_1(D_1^2(f_t|\Sigma(f_t))) = \mu(D_1^2(f_t|\Sigma(f_t))) + m_0(D_1^2(f_t|\Sigma(f_t))) - 1.$$

The ideal that defines the double points curve in the source is $I_1^2(f_t|\Sigma) = (xy + (a+t)y^3 + y^4, 2z^2 + x)$. As the second component of this ideal is regular, we can consider the double points curve embedded in $\mathbb{C}^2$, i.e. the following equalities hold:

$$\begin{aligned} \mu(D_1^2(f_t|\Sigma(f_t))) &= \mu((xy + (a + t)y^3 + y^4, 2z^2 + x)) \\ &= \mu(-z^2y + (a + t)y^3 + y^4) \\ &= 4, \text{ if } (a + t) \neq 0. \end{aligned}$$

So $\mu(D_1^2(f_t|\Sigma(f_t))) = \mu(D_1^2(f_0|\Sigma(f_0)))$, if $a \neq 0$ and for t small. We now calculate $m_0(f_t(D_1^2(f_t|\Sigma(f_t))))$. We have

$$2m_0(f_t(D_1^2(f_t|\Sigma(f_t)))) = e(I_1^2(f_t|\Sigma(f_t)), p \circ f_t)$$

where $p : \mathbb{C}^3 \to \mathbb{C}$ is a generic linear projection. As $(I_1^2(f_t|\Sigma(f_t)), p \circ f_t)$ is a reduced ideal and $\mathcal{O}_{D_1^2(f_t|\Sigma(f_t))}$ is Cohen Macaulay, $e(I_1^2(f_t|\Sigma(f_t)), p \circ f_t) = dim_{\mathbb{C}} \frac{\mathcal{O}_2}{(I_1^2(f_t|\Sigma(f_t)), p\circ f_t)}$. Taking $p(x, y, z) = x$, which is generic,

$$\begin{aligned} e(I_1^2(f_t|\Sigma(f_t)), p \circ f_t) &= dim_{\mathbb{C}} \tfrac{\mathcal{O}_3}{(xy+(a+t)y^3+y^4, 2z^2+x, x)} \\ &= 6 \text{ if } (a + t) \neq 0. \end{aligned}$$

Therefore

$$\begin{aligned} m_0(f_t(D^2_1(f_t|\Sigma(f_t)))) &= 3 \\ &= m_0(f_0(D^2_1(f_0|\Sigma(f_0)))), \text{ if } a \neq 0. \end{aligned}$$

Hence

$$\begin{aligned} m_1(D^2_1(f_t|\Sigma(f_t))) &= 6 \\ &= m_1(D^2_1(f_0|\Sigma(f_0))), \text{ if } a \neq 0 \text{ and } t \text{ small.} \end{aligned}$$

Finally we calculate the multiplicities of $f_t(\Sigma^{1,1}(f_t))$. The ideal that defines the stratum $\Sigma^{1,1}(f_t)$ is $I^{1,1}(f_t) = (4z^3 + xy + (a+t)y^3 + y^4 + 2xz, 12z^2 + 2x)$, so

$$m_0(f_t(\Sigma^{1,1}(f_t))) = dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(I^{1,1}(f_t), p \circ f_t)} = 3.$$

We can also calculate $\mu(\Sigma^{1,1}(f_t))$ as follows:

$$\begin{aligned} \mu(\Sigma^{1,1}(f_t)) &= \mu(4z^3 - 6z^2y + (a+t)y^3 + y^4 - 12z^3) \\ &= 4 \text{ if } (a+t) \neq 0, 1/2 \\ &= \mu(\Sigma^{1,1}(f_0)), \text{ if } a \neq 0,\ 1/2 \text{ and } t \text{ small.} \end{aligned}$$

As $\sharp A_3 = 3$ for $(a+t) \neq 0$, using Theorem 3.3.9, we have

$$m_1(f_t(\Sigma^{1,1}(f_t))) = m_1(f_0(\Sigma^{1,1}(f_0))) = 3, \text{ if } a \neq 0,\ 1/2 \text{ and } t \text{ small.}$$

We can now apply Theorem 3.3.14 to conclude that the family F is Whitney equisingulary for t sufficiently small if $a \neq 0, 1/2$.

In the same way, we show that the family G is also Whitney equisingulary if $a \neq 0, 1/2$.

□

Proposition 4.4.2. *Let $F(t,x,y,z) = (t,x,y,z^4 + (xy + ty^3 + (a+t)y^4)z + xz^2)$ be a unfolding of the germ $f(x,y,z) = (x,y,z^4 + (x^2 + xy^2 + ay^4)z + xz^2)$. Then F is Whitney equisingular for t sufficiently small if $a \neq 0, 1/4$.*

Proof: Let $f_t(x,y,z) = (x,y,z^4 + (x^2 + xy^2 + (a+t)y^4)z + xz^2)$ be a deformation of f. We have to calculate 6 invariants according to Theorem 3.3.14. We have

$$\begin{aligned} m_1(\Delta(f_t)) &= dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(y, 4z^3 + x^2 + xy^2 + (a+t)y^4 + 2xz, J[x, y, 4z^3 + x^2 + xy^2 + (a+t)y^4 + 2xz])} \\ &= 3 \\ &= m_1(\Delta(f_0)). \end{aligned}$$

As $\Sigma(f_t) = 4z^3 + x^2 + xy^2 + (a+t)y^4 + 2xz$ is a hypersurface with an isolated singularity, we have

$$\begin{aligned} \mu(\Sigma(f_t)) &= dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(2x + y^2 + 2z, 2xy + 4(a+t)y^3, 12z^2 + 2x)} = 3, \text{ if } (a+t) \neq 0 \\ &= \mu(\Sigma(f_0)), \text{ if } a \neq 0 \text{ and } t \text{ small.} \end{aligned}$$

Also,

$$m_1(\Sigma(f_t)) = \mu^2(\Sigma(f_t)) + \mu^1(\Sigma(f_t)).$$

Choose generic hyperplanes $H_1 = \{x = z = 0\}$ and $H_2 = \{y = 0\}$ so that

$$m_1(\Sigma(f_t)) = \mu^2(4z^3 + 2xz) + \mu^1(4x^2) = 2 = m_1(\Sigma(f_0)).$$

To calculate $m_1(D_1^2(f_t|\Sigma(f_t)))$ we needed to find the ideal that defines the curve $D_1^2(f_t|\Sigma(f_t))$ and use the following relation from Proposition 3.3.13

$$m_1(D_1^2(f_t|\Sigma(f_t)))) = \mu(D_1^2(f_t|\Sigma(f_t)))) + m_0(D_1^2(f_t|\Sigma(f_t)))) - 1.$$

The ideal that defines the double points curve in the source is $I_1^2(f_t|\Sigma(f_t)) = (x^2 + xy^2 + ty^4, 2z^2 + x)$. As the second component of this ideal is regular, we have

$$\begin{aligned}
\mu(D_1^2(f_t|\Sigma(f_t))) &= \mu(x^2 + xy^2 + (a+t)y^4, 2z^2 + x) \\
&= \mu(4z^4 - 2z^2y^2 + (a+t)y^4) \\
&= dim_{\mathbb{C}} \tfrac{\mathcal{O}_2}{(16z^3 - 4zy^2, -4z^2y + 4(a+t)y^3)} \\
&= dim_{\mathbb{C}} \tfrac{\mathcal{O}_2}{(z, -4z^2y + 4(a+t)y^3)} + dim_{\mathbb{C}} \tfrac{\mathcal{O}_2}{(16z^2 - 4y^2, -4z^2y + 4(a+t)y^3)} \\
&= 3 + dim_{\mathbb{C}} \tfrac{\mathcal{O}_2}{(16z^2 - 4y^2, -4z^2y + 4(a+t)y^3)} \\
&= 3 + dim_{\mathbb{C}} \tfrac{\mathcal{O}_2}{((4z-2y)(4z+2y), -4z^2y + 4(a+t)y^3)} \\
&= 9, \text{ if } a + t \neq 0, 1/4 \\
&= \mu(D_1^2(f_0|\Sigma(f_0)))), \text{ if } a \neq 0, 1/4, \text{and } t \text{ small.}
\end{aligned}$$

To calculate $m_0(D_1^2(f_t|\Sigma(f_t)))$ it is enough to use the definition of the multiplicity, that is,

$$m_0(D_1^2(f_t|\Sigma(f_t))) = dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(x^2 + xy^2 + (a+t)y^4, 2z^2 + x, y)} = 4.$$

Hence

$$m_1(D_1^2(f_t|\Sigma(f_t))) = 12 = m_1(D_1^2(f_0|\Sigma(f_0))), \text{ if } a \neq 0, 1/4 \text{ and } t \text{ small.}$$

For the multiplicity $m_0(f_t(D_1^2(f_t|\Sigma(f_t))))$ we have

$$2m_0(f_t(D_1^2(f_t|\Sigma(f_t)))) = e(I_1^2(f_t|\Sigma(f_t)), p \circ f_t)$$

where $p : \mathbb{C}^3 \to \mathbb{C}$ is a generic linear projection. As $(I_1^2(f_t|\Sigma(f_t)), p \circ f_t)$ is a reduced ideal and $\mathcal{O}_{D_1^2(f_t|\Sigma(f_t))}$ is Cohen-Macaulay,

$$e(I_1^2(f_t|\Sigma(f_t)), p \circ f_t) = dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(I_1^2(f_t|\Sigma(f_t)), p \circ f_t)}.$$

Choosing $p(x, y, z) = y$ which is generic,

$$e(I_1^2(f_t|\Sigma(f_t)), p \circ f_t) = dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(x^2 + xy^2 + (a+t)y^4, 2z^2 + x, y)} = 4.$$

Therefore,

$$m_0(f_t(D^2_1(f_t|\Sigma(f_t)))) = 2 = m_0(f_0(D^2_1(f_0|\Sigma(f_0)))).$$

Now for the multiplicity of $f_t(\Sigma^{1,1}(f_t))$, the ideal that defines the stratum $\Sigma^{1,1}(f_t)$ is $I^{1,1}(f_t) = (4z^3 + x^2 + xy^2 + (a+t)y^4 + 2xz, 12z^2 + 2x)$. So

$$\begin{aligned} m_0(f_t(\Sigma^{1,1}(f_t))) &= dim_{\mathbb{C}} \tfrac{\mathcal{O}_3}{(I^{1,1}(f_t), p\circ f_t)} \\ &= dim_{\mathbb{C}} \tfrac{\mathcal{O}_3}{(4z^3+x^2+xy^2+(a+t)y^4+2xz, 12z^2+2x, y)} \\ &= dim_{\mathbb{C}} \tfrac{\mathcal{O}_2}{(4z^3+x^2+2xz, 12z^2+2x)} \\ &= 3 \\ &= m_0(f_0(\Sigma^{1,1}(f_0))). \end{aligned}$$

We calculate $\mu(\Sigma^{1,1}(f_t))$ in the same way as in the previous proposition. We have $\mu(\Sigma^{1,1}(f_t)) = 6$ for $(a+t) \neq 0$ and $\sharp A_3 = 3$ for $(a+t) \neq 0$. Then by Theorem 3.3.9, we have $m_1(f_t(\Sigma^{1,1}(f_t))) = m_1(f_0(\Sigma^{1,1}(f_0))) = 3$ for $a \neq 0,\ 1/4$ and t small.

We apply now Theorem 3.3.14 to assert that the family F is Whitney equisingular for t sufficiently small and $a \neq 0, 1/4$. □

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